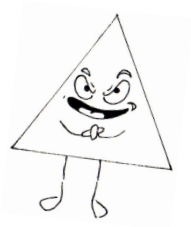


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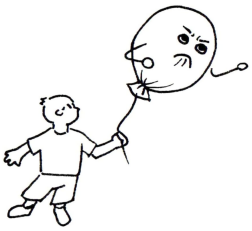
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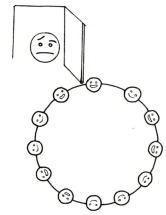
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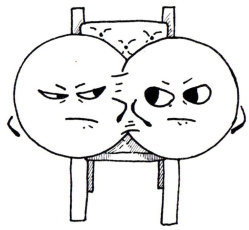




Olympiad physics from Kvant *and Russian olympiads, translated into English and annotated*

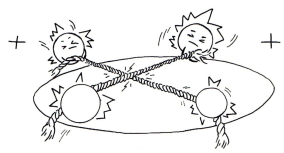




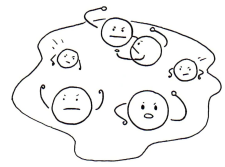












Text and translation: Ivan Rudik Illustrations: Veronika Lukanovskaya

Version 0.9 · November 2026

The Other Problem Book
Olympiad physics from *Kvant* and Russian olympiads, translated into English and annotated

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I

PART I

Problems 1–10

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6 Gas Circle
7 Sealed Tube with Mercury
8 Balloon with a Vent Hole
9 Chain
10 Two Balls

PROBLEM 1

Partially Charged Cylinder

From the All-Russian Olympiad, final stage, 2021

Veronika's art
p01.png

Problem. An infinitely long, thin-walled dielectric cylinder of radius R is divided, along the axis of rotation, into equal "quarters." Two opposite quarters are uniformly charged with surface charge density $\sigma > 0$, and the other two are left uncharged.

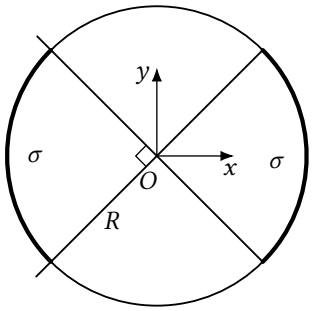


Fig. 1.1

Find the electric field vectors of the cylinder at points close to its center having coordinates $(x, 0)$ and $(0, y)$. Assume $x, y \ll R$. The coordinate axes OX and OY are directed along the bisectors of the right angles (see Fig. 1.1).

Solution. The first question that arises in this problem is how to calculate the electric field at a given point. To do this, we break the charged surface into small pieces and calculate the field of each piece separately.

The field of the cylinder can be represented as a sum of the fields of infinitely long thin threads (very narrow strips) that are charged with constant linear charge density. We find the field of such a thread at the distance r from it, using the symmetry of the field and Gauss's law:

$$\Phi = 2\pi r L\, E_r = \frac{q}{\varepsilon_0} = \frac{\lambda L}{\varepsilon_0}, \qquad E_r = \frac{\lambda}{2\pi\varepsilon_0 r}.$$

We surrounded the thread with a coaxial cylinder of radius r and took into account the flux only through its lateral surface.

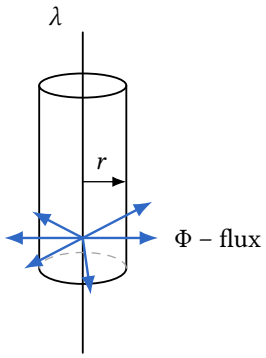


Fig. 1.2

FIELD AT THE POINT $(x, 0)$

Now let us find the field at the point $(x, 0)$. Consider two small sections dL_1 and dL_2 of the cylinder highlighted in Fig. 1.3.

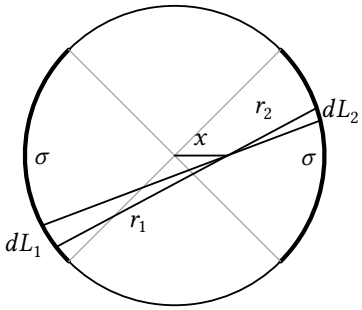


Fig. 1.3

They create oppositely directed fields as from two line charges with linear densities $\sigma\, dL_1$ and $\sigma\, dL_2$, respectively. From the similarity of triangles we have

$$\frac{dL_1}{r_1} = \frac{dL_2}{r_2},$$

therefore

$$dE_1 = \frac{\sigma\, dL_1}{2\pi\varepsilon_0 r_1}, \qquad dE_2 = \frac{\sigma\, dL_2}{2\pi\varepsilon_0 r_2},$$

and these two fields cancel each other.

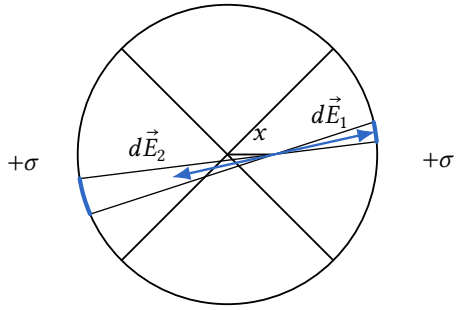


Fig. 1.4

No matter which part we take on the left side, there will always be a part on the right side that compensates for the one on the left. However, when considering the extreme points on the left, it turns out that strips of width dS remain, whose fields are not compensated by anything.

It follows that the field of the cylinder at the point under consideration is equivalent to the total field of two selected strips of width dS each, for which the fields do not cancel (see Fig. 1.5).

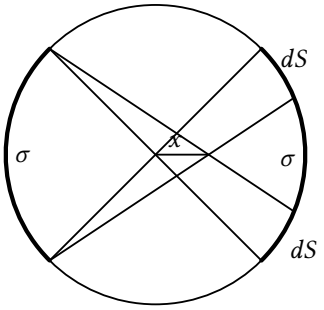


Fig. 1.5

At the same time

$$dS = x\sqrt{2},$$

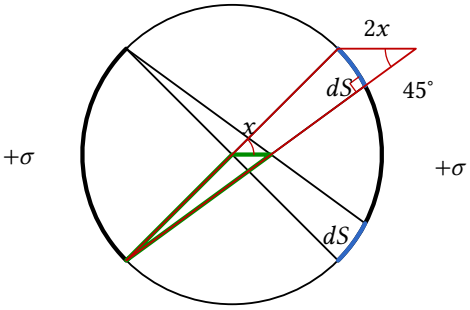


Fig. 1.6

Why is that the case?

We can say that due to the smallness of x and dS, the lines are close to each other. From the similarity of the green and red triangles, we see that the larger side is equal to $2x$. Furthermore, the angle between x (the horizontal) and the line is 45° because the circle is divided into quarters. Thus, we obtain a right triangle with the hypotenuse $2x$, the leg dS, and the angle of 45° (due to the parallelism, the properties of the right triangle, and the fact that all angles are 45°).

From here, we get:

$$dS = 2x \sin(45^\circ) = 2x\frac{\sqrt{2}}{2} = x\sqrt{2}$$

and each strip produces a field equal to

$$E_0 = \frac{\sigma x\sqrt{2}}{2\pi R\,\varepsilon_0}.$$

The fields of these strips cancel each other in the y-direction, hence $E_y = 0$, and the projection of the field on the x-axis is

$$E_x = -\frac{2E_0}{\sqrt{2}} = -\frac{\sigma x}{\pi R\,\varepsilon_0}.$$

Thus, the resulting field is directed toward the cylinder axis.

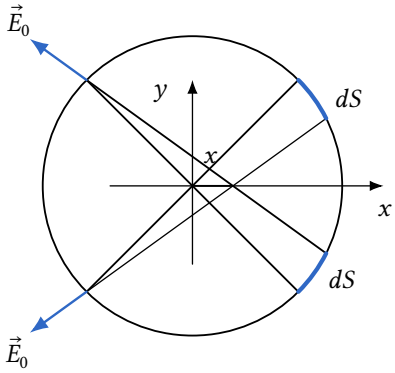


Fig. 1.7

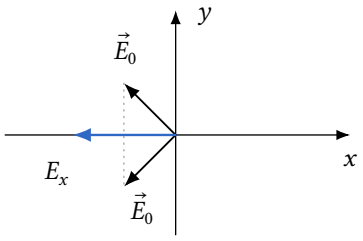


Fig. 1.8

FIELD AT THE POINT $(0, y)$

To find the field at $(0, y)$, represent the original cylinder as a superposition of a fully charged cylinder with surface charge density $+\sigma$ and two quarters with surface density $-\sigma$ (see Fig. 1.10).

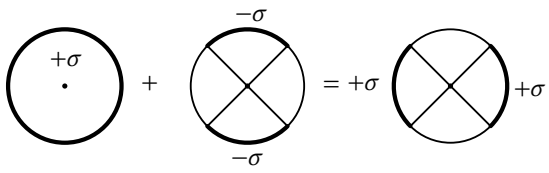


Fig. 1.9

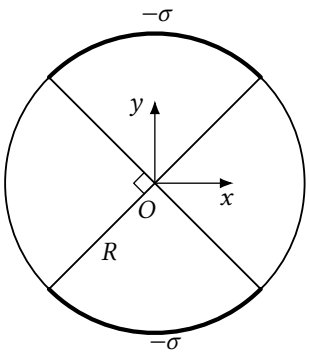


Fig. 1.10

The field E_1 created by the two negatively charged quarters is analogous to that found in the first case, so we can use the ready result:

$$E_{1y} = \frac{\sigma y}{\pi R \varepsilon_0}, \qquad E_{1x} = 0.$$

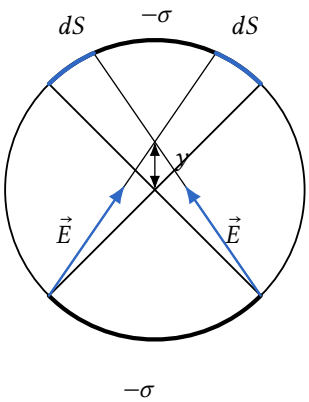


Fig. 1.11

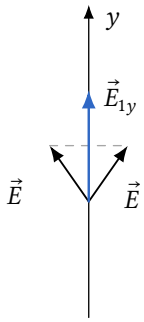


Fig. 1.12

The field created at the point $(0, y)$ by a uniformly charged cylinder is zero by symmetry (all parts can be split into pairs whose fields cancel) and Gauss's law. Then, by the principle of superposition, the projections of the resulting field E' at the desired point are:

$$E'_y = \frac{\sigma y}{\pi R\, \varepsilon_0}, \qquad E'_x = 0.$$

The field E' is directed away from the cylinder axis.

PROBLEM 2

From the Hole onto the Field

From the All-Russian Olympiad, final stage, 2023

Veronika's art
p02.png

Problem. There is a hole in a playing field, from which a small ball is ejected. Immediately before emerging, the speed of the ball equals v_0 and is directed vertically upward. At the instant of emergence the ball collides *perfectly elastically* with another identical ball, which prior to impact was at rest at the level of the field surface.

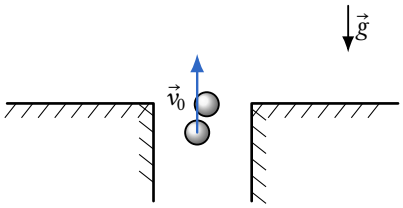


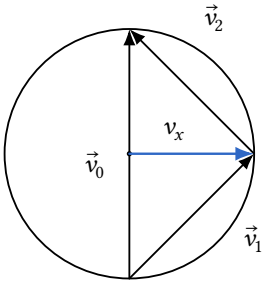
Fig. 2.1

After the collision the balls move only under gravity and land somewhere on the field. The balls' impacts with the field are *perfectly inelastic*.

1. Find the maximum possible distance $l_{\max}$ between the landing points of the balls.
2. Find the maximum possible distance $S_{\max}$ from the hole to the landing point of one of the balls.

Assume the field surface is horizontal. The sizes of the hole and of the balls are small compared with $l_{\max}$ and $S_{\max}$.

Solution.



Max when $v_x = v_0/2$

Fig. 2.2

1.

We use conservation of momentum and of energy for the collision:

$$\begin{cases} m\vec{v}_0 = m\vec{v}_1 + m\vec{v}_2 & \text{(momentum)} \\ \dfrac{mv_0^2}{2} = \dfrac{mv_1^2}{2} + \dfrac{mv_2^2}{2} & \text{(energy)} \end{cases}$$

Squaring the momentum equation we obtain

$$v_0^2 = v_1^2 + v_2^2 + 2\,\vec{v}_1 \cdot \vec{v}_2.$$

Together with energy conservation ($v_0^2 = v_1^2 + v_2^2$) this gives

$$2\,\vec{v}_1 \cdot \vec{v}_2 \Rightarrow \vec{v}_1 \perp \vec{v}_2.$$

For convenience we use a velocity diagram, a standard tool in kinematics problems.

Since $v_1^2 + v_2^2 = v_0^2$ and $\vec{v}_0 = \vec{v}_1 + \vec{v}_2$, the velocity triangle is right-angled and is inscribed in a circle with diameter v_0.

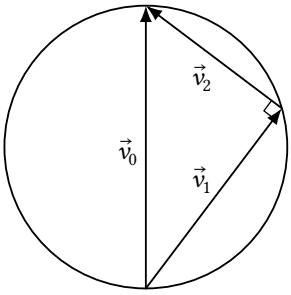


Fig. 2.3

Introduce at the launch point the coordinate system yOx, where y is vertical upward and x is horizontal. Below, $v_{x(i)}$ and $v_{y(i)}$ denote the magnitudes of the horizontal and vertical components of the velocity of the i-th ball ($i = 1, 2$) immediately after the collision. Note that $v_{x(1)} = v_{x(2)} = v_x$; the vertical velocity at landing is $-v_{y(i)}$, so $gt_i = v_{y(i)} + v_{y(i)} = 2v_{y(i)}$.

Find the range of the i-th ball:

$$t_i = \frac{2v_{y(i)}}{g}, \qquad l_i = v_x t_i \;\Rightarrow\; l_i = \frac{2v_x v_{y(i)}}{g}.$$

For the distance between the landing points we have

$$l = l_1 + l_2 = \frac{2v_x\left(v_{y(1)} + v_{y(2)}\right)}{g} = \frac{2v_0 v_x}{g}.$$

Thus, the distance between the landing points is maximal for the maximal value of v_x, equal to the radius of the circle, i.e. $v_0/2$.

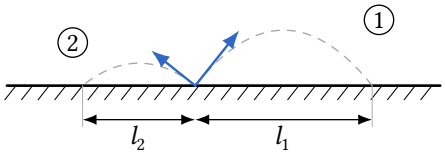


Fig. 2.4

Therefore, the answer to the first question is

$$l_{\max} = \frac{v_0^2}{g}.$$

2.

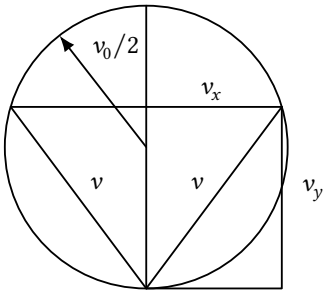


Fig. 2.5

The maximal range of one of the balls corresponds to the maximal value of the product $v_x v_{y(i)}$, which is therefore the same for both balls. Geometrically, the product $v_x v_y$ equals the area of a rectangle with sides v_x and v_y. In turn, this area equals the area of an isosceles triangle with equal sides

$$v = \sqrt{v_x^2 + v_y^2}$$

and base $2v_x$. This triangle is inscribed in a circle of radius $v_0/2$. The maximal area of a triangle inscribed in a circle of radius R is attained for an equilateral triangle and equals

$$S = \frac{3\sqrt{3}\,R^2}{4} \quad\Longrightarrow\quad (v_x v_y)_{\max} = \frac{3\sqrt{3}\,v_0^2}{16},$$

whence

$$l_{i(\max)} = \frac{3\sqrt{3}\,v_0^2}{8g}.$$

PROBLEM 3

Remembering the 1990s

From the All-Russian Olympiad, regional stage, 2024

Veronika's art
p03.png

Problem. Recalling 1998, the theoretician Bag continued to study the following situation. A board a lies on an identical board b. As a single unit, they move with some speed along a smooth horizontal surface and collide with an identical board c, whose upper surface is covered with a thin layer of rubber. There is no friction between boards a and b, while there *is* friction between boards a and c. At the impact, boards b and c stick firmly together. The length of each board is L. Board a ceases to move relative to boards b and c at the instant when it is entirely located on top of board c. Find the displacement Δx of board a from the instant of impact until the cessation of the relative motion of the boards.

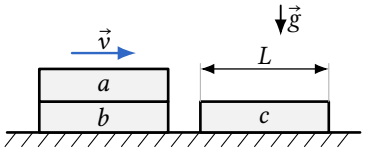


Fig. 3.1

Solution. Let v_0 be the speed of boards a and b before colliding with board c, and m the mass of each board. Since there is no friction between a and b, and boards b and c stick together at the impact, right after the impact the speed of board a is v_0, and the speed of boards b and c is $v_0/2$ (by the law of conservation of momentum).

Introduce the x-axis along the direction of the velocity v_0. Since the horizontal surface is smooth, the center of mass of the system moves with constant speed $v_{\text{CM}} = \frac{mv_0+mv_0+0}{m+m+m} = \frac{2v_0}{3}$.

At the instant of impact the center of mass of the system is ahead of the center of board a by $L/3$, and by the moment when board a is entirely on board c the center of mass lags behind the center of the board by the same amount $L/3$.

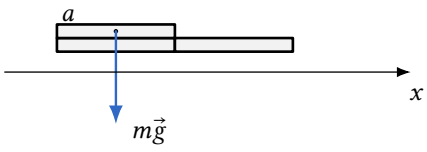


Fig. 3.3

Sketch to be added
ДОБАВИТЬ КАРТИРИНКИ БОЛЕЕ ПОДРОБНЫЕ
ЧТОБЫ БЫЛО ПОНЯТНО КАК НАЙДЕН ЦМ

Fig. 3.2

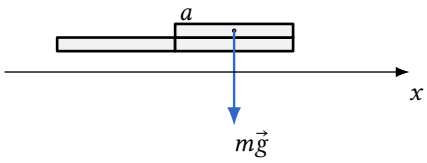


Fig. 3.4

$$x_{\mathrm{CM}} = \frac{\frac{L}{2} \times 2m + \frac{3}{2}Lm}{3m} = \frac{L + \frac{3}{2}L}{3} = \frac{5}{6}L \qquad \frac{5}{6}L - \frac{L}{2} = \frac{2}{6}L = \frac{1}{3}L$$

That is, the center of mass of the system is ahead of the center of board a by $\frac{L}{3}$

$$x'_{\mathrm{CM}} = \frac{\frac{3}{2}L \times 2m + m\frac{L}{2}}{3m} = \frac{3L + \frac{L}{2}}{3} = \frac{7}{6}L \qquad \frac{3}{2}L - \frac{7}{6}L = \frac{1}{3}L$$

That is, the center of mass of board a is ahead of the center of mass of the system by $\frac{L}{3}$

Therefore, for the displacement of board a we obtain

$$\Delta x = \Delta x_{\mathrm{CM}} + \frac{2L}{3} = \frac{2\,(v_0\tau + L)}{3},$$

(That is, board a moves by $\frac{2L}{3}$ relative to the center of mass of the system, and the center of mass itself moves by Δx_{CM}.)

where τ is the time of motion of board a *relative* to boards b and c. Next, combine boards b and c into a single body of mass $2m$ and designate it by index 2. Board a will be designated by index 1. Let $\vec{F}$ be the friction force acting on the first body from the second, and let x be their relative displacement after the impact. Note that friction acts only in the overlap region of boards a and c. The kinetic-friction force is directly proportional to the normal reaction of the rough part of the support surface, and therefore

$$F_x = -\mu m g \frac{x}{L}.$$

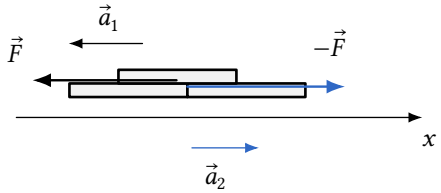


Fig. 3.5

Write Newton's second law for each body:

$$m\vec{a}_1 = \vec{F}, \qquad 2m\vec{a}_2 = -\vec{F}.$$

Let us find the relative acceleration:

$$\vec{a}_1 - \vec{a}_2 = \vec{a}_r = \frac{\vec{F}}{m} + \frac{\vec{F}}{2m} = \frac{3\vec{F}}{2m} = \ddot{x}$$

Then we obtain the equation of motion for the variable x:

$$\ddot{x} = -\frac{3\mu g}{2L}\, x.$$

This is the equation of harmonic oscillations with circular frequency $\omega = \sqrt{3\mu g/2L}$. Its general solution is

$$x(t) = A\sin(\omega t + \varphi_0).$$

Determine A and φ_0 from the initial conditions:

$$\begin{cases} x(0) = A\sin\varphi_0 = 0, & (1) \\ \dot{x}(0) = \dfrac{v_0}{2} = \omega A\cos\varphi_0 & (2) \end{cases} \quad \Rightarrow \quad \begin{cases} \varphi_0 = 0, \\ A = \dfrac{v_0}{2\omega}. \end{cases}$$

1) since at the start board a has not yet moved onto board c;

2) since $v_0 - \frac{v_0}{2}$ is the speed with which board a slides onto board c at the start.

The quantity A is the oscillation amplitude, which is also equal to L, since the relative motion of the boards stops when board a is entirely on board c. Thus we have the relation

$$\frac{v_0}{2\omega} = L.$$

The time of motion τ equals a quarter of the period of harmonic oscillations with circular frequency ω, hence

$$\tau = \frac{\pi}{2\omega}.$$

(A full period $T = \frac{2\pi}{\omega}$ consists of four such quarter-swings, and only one of them occurs here.)

Then, for the displacement of the center of mass we have

$$\Delta x_{\mathrm{CM}} = \frac{2v_0\tau}{3} = \frac{2(2\omega L)\times(\frac{\pi}{2\omega})}{3} = \frac{2\pi L}{3},$$

and finally we find

$$\Delta x = \Delta x_{\mathrm{CM}} + \frac{2L}{3} = \frac{2(\pi+1)L}{3}.$$

PROBLEM 4

Fan Thrust

From Kvant, *1979, No. 10*

Veronika's art
p04 . png

Problem. A fixed (mounted) air fan consumes power N and has efficiency η. What reactive force (thrust) acts on the fan while it operates? The blade diameter is D, and the air density is ρ.

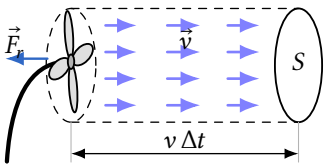


Fig. 4.1

Solution. Let v denote the magnitude of the velocity of the air stream created by the fan. Then the mass of air set in motion in a time Δt is

$$\Delta m = \rho v S \Delta t,$$

where

$$S = \frac{\pi D^2}{4}$$

is the cross-sectional area of the air stream (see Fig. 4.1), and its kinetic energy is

$$E_{\text{kin}} = \frac{mv^2}{2} = \frac{\rho v^3 S \Delta t}{2}.$$

Efficiency is the ratio of useful work to the work expended. The useful work in this case is the kinetic energy given to the air flow, while the expended work is the energy consumed by the fan. So,

$$\eta = \frac{E_{kinetic}}{Nt}$$

Therefore, the power of the fan is

$$N = \frac{E_{\text{kin}}}{\eta t} = \frac{\rho v^3 S}{2\eta}.$$

From this one can find the air-stream speed:

$$v = \left(\frac{2\eta N}{\rho S}\right)^{1/3},$$

and the reactive force acting on the fan:

$$F_r = \rho S v^2 = (\rho S)^{1/3}(2\eta N)^{2/3} = \left(\pi\rho\right)^{1/3}(\eta D N)^{2/3}.$$

(Recall that the reactive force (thrust) is $F_r = \frac{\Delta m}{\Delta t} v$)

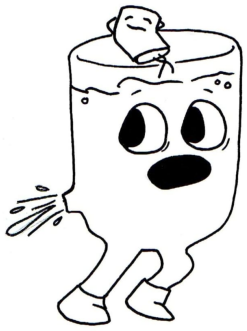
If a mechanical system moves at a constant speed so that the change of its momentum occurs only due to the ejection of mass, then to find the reactive force it is convenient to switch to the reference frame attached to the moving body.

In this problem, since the fan is fixed, we are already in the reference frame of the fan.

PROBLEM 5

Vessel with Water

From Kvant, *1979, No. 10*



Problem. On a horizontal surface stands a vessel filled with water, closed on top by a light movable piston bearing a heavy load of mass M (see Fig. 5.1). There is an orifice of cross-section s at the bottom of the vessel through which water flows out. What steady speed of motion of the vessel will be established if the friction force between the vessel and the surface is proportional to the vessel's speed, the proportionality coefficient being k? The cross-sectional area of the vessel is S.

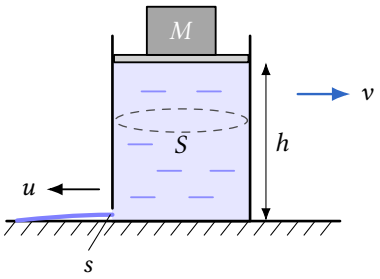


Fig. 5.1

Solution. Just like in Problem 4, we will use the formula for the reactive force here as well. Let's recall what reactive force is:

$$F_r = u\frac{\Delta m}{\Delta t}$$

That is, each element of the fluid carries away a momentum $\Delta m\, u$, thereby creating an impulse of force $F_r \Delta t$ that pushes the vessel forward.

The desired speed (more precisely, its magnitude v) is determined from the condition that the magnitudes of the friction force and the reactive force acting on the vessel are equal:

$$F_{\text{fr}} = kv = F_r.$$

To find the reactive force, pass to the reference frame moving with speed v. In this frame the vessel is at rest. To determine the outflow speed we use energy conservation.

When a small amount of water of mass Δm flows out, the change of kinetic energy of

the water $\Delta m\, u^2/2$ equals the change of potential energy of the water Δmgh (here h is the water level in the vessel) and of the load $Mg\Delta h$ (where Δh is the displacement of the load, equal to the change of the water level):

$$\frac{\Delta m\, u^2}{2} = \Delta mgh + Mg\Delta h.$$

The liquid flowing out of the vessel pushes the upper layers of water and the piston downward. Because this process is quite complex, calculating the total energy is difficult. To estimate the potential energy, we will assume that all the lower layers remain stationary and only the upper layer of water exits the vessel. Thus, the potential energy of the upper layer and the piston is converted into the kinetic energy of the outflowing water.

The change Δh of the water level in the vessel can be expressed through the mass Δm of the outflowing water: $\Delta h = \Delta m/(\rho S)$. Substituting this relation into the energy equation and canceling Δm, we obtain for the outflow speed u

$$u = \sqrt{2gh + \frac{2Mg}{\rho S}}.$$

If the load is sufficiently heavy (that is, its mass is much greater than the mass of water in the vessel: $M \gg \rho Sh$), the first term under the root may be neglected. In this case the outflow speed is practically constant and equals

$$u = \sqrt{\frac{2Mg}{\rho S}}.$$

Now we can find the reactive force acting on the vessel:

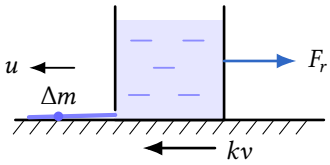


Fig. 5.2

$$F_p = u\frac{\Delta m}{\Delta t} = u\frac{\rho su\,\Delta t}{\Delta t} = \rho su^2 = \frac{2Mg\,s}{S},$$

and the speed of motion of the vessel:

$$v = \frac{F_p}{k} = \frac{2Mg\,s}{kS}.$$

PROBLEM 6

Gas Circle

From Kvant, *1972, No. 5*

Veronika's art
p06 . png

Problem. A certain process occurring with an ideal gas is represented on the graph of pressure versus temperature as a circle. It is required to divide the circle, with the smallest possible number of points, into parts corresponding to processes in each of which every one of the quantities P, V, and T changes monotonically (that is, either increases or decreases).

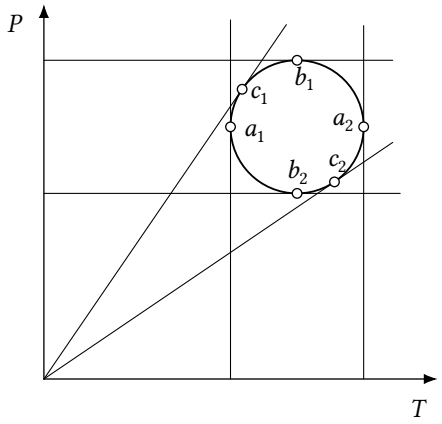


Fig. 6.1

Solution. First, we need to identify the critical points so that each of the quantities is monotonic—either increasing or decreasing—on the interval between those points. It's easy to see these are the maxima and minima, since between them the function either rises or falls. Therefore, we need to find the minimum and maximum values of P, T, and V.

The points corresponding to the highest and the lowest temperature in the given process are the points a_1 and a_2, where the circle is tangent to vertical isotherms (see Fig. 6.1). The points at which the pressure attains its maximum and minimum are the points b_1 and b_2, where the circle is tangent to horizontal isobars.

Since this is an ideal gas with a constant mass, from the equation of state (or the Clapeyron–Mendeleev equation, $PV = nRT$), we conclude that each value of V is represented by an inclined line ($V = \frac{nRT}{P}$) on the $P(T)$ graph.

So, isochores on the $P(T)$ graph are depicted by straight lines passing through the origin of coordinates. The smaller the volume, the steeper the isochore. Since the slope on the graph represents the ratio $\frac{P}{T}$, and the volume V is inversely proportional to this ratio, we have that $V \propto \frac{T}{P}$.

Thus, to find $V_{\max}$ and $V_{\min}$, we need to take the lines with the largest and smallest slopes passing through the origin — in this case, these will be the tangents to the circle, c_1 and c_2.

Altogether, the circle is thereby divided into six arcs, each arc representing one of the required processes.

From what has been said it is clear that the parameters determining the state of a given mass of gas of unchanging chemical composition (P, V, T) are related to each other. If two quantities are known, the third can be found.

PROBLEM 7

Sealed Tube with Mercury

From Kvant, *1990, No. 8*

Veronika's art
p07 . png

Problem. The lower end of a vertical narrow tube of length $2L$ (in mm) is sealed, and the upper end is open to the atmosphere. In the lower half of the tube there is a gas at temperature T_0, and the upper half of the tube is filled with mercury. The tube is slowly heated. To what *minimum* temperature must the gas in the tube be heated so that it expels all the mercury? The external pressure p_0, measured in millimeters of mercury, is equal to L.

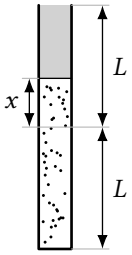


Fig. 7.1

Solution. First, let us find at what temperature T the gas in the tube will expel a column of mercury of length x (see Fig. 7.1). To this end we write the combined gas law together with the equation of hydrostatics:

$$\frac{(p_0+\rho g L)\,L}{T_0}=\frac{p(L+x)}{T},\qquad p=p_0+\rho g(L-x).$$

(In the first equation the left-hand side refers to the initial state, and the right-hand side to the state in which the gas has pushed out a column of length x.)

Taking into account that $p_0=\rho g L$, we obtain

$$T=T_0\left(1-\frac{x}{2L}\right)\left(1+\frac{x}{L}\right).$$

If we try to obtain the answer to the problem by substituting $x=L$ into this formula, we get $T=T_0$. In other words, the gas would not have to be heated at all. But this is not so. The point is that the function $T(x)$ is not monotonic (see Fig. 7.2), and at $x=L/2$ it has a

maximum equal to

$$T_{\max} = \frac{9}{8}\, T_0$$

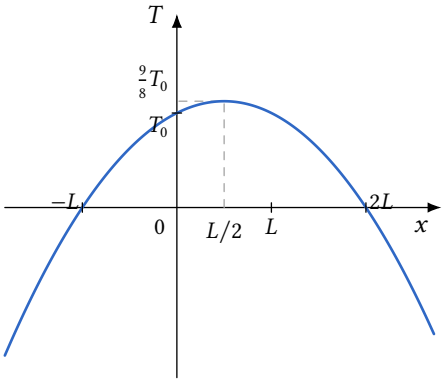


Fig. 7.2

(to obtain this, recall that the vertex of a parabola lies halfway between its roots, which here are, obviously, $x_1 = 2L$ and $x_2 = -L$). Thus, to expel all the mercury from the tube it is sufficient to heat it to a temperature $T > T_{\max}$, at which the equilibrium of the mercury becomes unstable, and thereafter everything proceeds by itself.

PROBLEM 8

Balloon with a Vent Hole

From Kvant, *1975, No. 1*

Veronika's art
p08.png

Problem. An inextensible shell of a sounding balloon has volume $V = 75$ m^3 and a small opening at its lower part. The shell mass is $m = 7$ kg. The balloon is filled with hydrogen. Determine the maximum altitude to which this balloon can rise if the atmospheric pressure is known to decrease by a factor of two for every $h = 5$ km in height. The air temperature in the stratosphere is $t = -60^\circ$C; the hydrogen temperature equals the ambient air temperature. The pressure at the Earth's surface is $p_0 = 1$ atm.

Sketch to be added
ВСТАВИТЬ КАРТИНКУ

Fig. 8.1

Solution. Because the shell of the balloon is not airtight and has an opening, the pressure inside the balloon is always equal to the atmospheric pressure, and as the altitude increases, hydrogen flows out through the opening. Since the volume and temperature of the balloon remain constant while the pressure decreases with altitude, according to the ideal-gas law ($pV = \mu RT$, so $\mu = \frac{pV}{RT}$) the amount of substance must decrease. We shall assume that the ascent is sufficiently rapid so that diffusion (penetration) of air into the shell can be neglected. Then the equilibrium condition for the balloon at the maximum altitude is

$$mg + \rho_{H_2}\, gV = \rho_{\text{air}}\, gV.$$

The gravitational forces of the shell and the hydrogen inside it are balanced by the buoyant (Archimedes) force in the air.

The densities of hydrogen and air can be obtained from the Clapeyron–Mendeleev equation:

$$\rho_{H_2} = \frac{\mu_{H_2}\, p}{RT}, \qquad \rho_{\text{air}} = \frac{\mu_{\text{air}}\, p}{RT}.$$

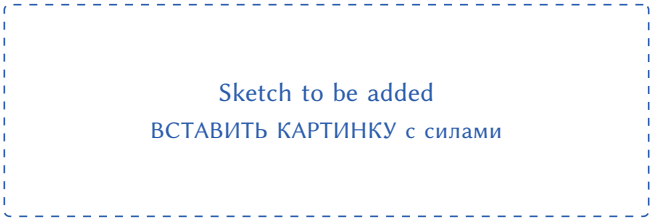


Fig. 8.2

Thus, the pressure at the maximum altitude is

$$p = \frac{mRT}{(\mu_{\text{air}} - \mu_{\text{H}_2})V} \approx 6.12{\times}10^3 \text{ N/m}^2.$$

The ratio $\frac{p_0}{p} = 16$, and therefore the ascent height is

$$H = 20 \text{ km}$$

Since the atmospheric pressure decreases by a factor of two for every $h = 5$ km in height, we can write:

$$\frac{p_0}{p} = 2^{\frac{H}{h}} = 16 \Rightarrow H = 4h = 20\,km$$

Let us now change the condition slightly.

What will happen if the balloon is airtight? To what altitude will it be able to rise under the same conditions?

The buoyant force will be the same for both balloons, since their volumes are equal. If the initial masses of the balloons were the same, then after ascent the balloon with the opening will be lighter, because part of the gas filling it flows out during the ascent. Therefore, the balloon with the opening will be able to rise to a greater height.

Usually, to someone who first thinks about this question, such a result seems strange. A common question is: "How does a lifting force arise at all in a balloon with an opening? After all, at the bottom, where the opening is, the air and the gas inside the balloon are in equilibrium."

Let us consider the top point of the balloon. If at the bottom point of the balloon the pressure of the air and the gas is p_0, then at the top point the air pressure is

$$p_1 = p_0 - \rho_{\text{air}}\, gh \quad (h \text{ is the height of the balloon}),$$

and the gas pressure is

$$p_2 = p_0 - \rho_g\, gh \quad (h \text{ is the height of the balloon}).$$

If $\rho_g < \rho_{\text{air}}$, then $p_2 > p_1$, and, consequently, a larger force acts on the shell from below than from above—a lifting force arises. It is easy to verify (you can do this yourself for a body of sufficiently simple shape) that it is precisely this difference in pressures that gives the resultant buoyant force determined by Archimedes' law.

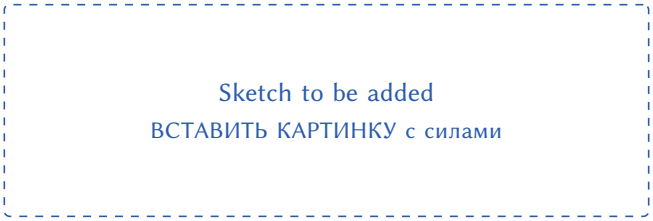


Fig. 8.3

A common mistake in calculations is that the pressure inside the balloon is usually taken to be the same everywhere. One should not forget that this is only an approximation. If we determine the absolute value

$$p_2 = p_0 - \rho_g\, gh,$$

then, since h is small—only a few meters—we have $\rho_{\text{air}}\, gh \ll p_0$, and we may set $p_2 \approx p_0$. If, however, we are interested in the difference

$$p_2 - p_1 = \rho_{\text{air}}\, gh - \rho_g\, gh,$$

then here both terms are of the same order of magnitude, and both must be taken into account. By the way, the assumption that we take ρ_{air} and ρ_g as constants is also an approximation; in reality they decrease with altitude as the pressure decreases. But taking this circumstance into account would yield a much smaller correction to the buoyant force, and this correction may be neglected.

Remark.

The density as a function of height in the troposphere is given by the following formula:

$$\rho = \rho_0 e^{-\left(\frac{gMh}{RT_0} - \frac{Lh}{T_0}\right)}$$

where

- p_0, sea level standard atmospheric pressure, 101325 Pa
- T_0, sea level standard temperature, 288.15 K
- g, earth-surface gravitational acceleration, 9.80665 m/s^2
- L, temperature lapse rate, 0.0065 K/m
- R, ideal (universal) gas constant, 8.31446 J/(mol·K)
- M, molar mass of dry air, 0.0289652 kg/mol

PROBLEM 9

Chain

From Kvant, *1980, No. 9*

Veronika's art
p09.png

Problem. There is a chain made of n identical, massless links connected by frictionless hinges (see Fig. 9.1). Neglecting friction, determine the tension that must be sustained by the string connecting points 1 and 2 if a load of mass m is suspended from the chain.

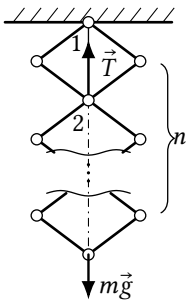


Fig. 9.1

Solution. We use the method of virtual displacements. Let us recall it.

Principle of virtual displacements

- For a mechanical system with ideal constraints to be in equilibrium, it is necessary and sufficient that the sum of elementary works of *active forces* $\vec{F}_i$ on any virtual displacement of the system be zero

$$\sum \vec{F}_i \delta\vec{r}_i = 0$$

- If some forces are conservative, then their work is equal to $-dE_p$, so

$$\sum \vec{F}_i \delta\vec{r}_i = dE_p$$

Imagine that the load of mass m is lowered by a distance δh. By symmetry it is clear that, in doing so, the long diagonal of each link increases in length by $\delta h/n$. Therefore, the virtual extension of the string between points 1 and 2, which is equal to the virtual displacement of point 2, is also $\delta h/n$.

The forces that arise in the joints will cancel each other when we consider the chain as a whole. Thus, the external forces acting on the chain will be: the tension forces of the string and the gravitational force of the load.

Write the condition that the total virtual work is zero:

$$T \cdot \frac{\delta h}{n} - mg \cdot \delta h = 0,$$

where T is the tension in the string. Hence,

$$T = nmg.$$

PROBLEM 10

Two Balls

From Kvant, *1975, No. 1*



Problem. Find the deformation that arises in spheres of radius R when they are subjected to two compressive forces F directed along the line connecting the centers of the spheres. The material of the spheres is identical, and its Young's modulus is E.

Solution. This problem will use the method of dimensional analysis. It allows us to derive a physically correct formula, up to a constant, knowing only the dimensions of the individual quantities. It is also very important to know what the quantity we want to find depends on.

To solve the problem, we need to find the dependence of the deformation x on the quantities F, R, and E. We seek this dependence in the form

$$x = A_3 F^n R^s E^t, \tag{1}$$

(where A_3 is a dimensionless constant. With this method we can "guess" the formula up to an unknown constant, which is often determined by carrying out a real experiment.)

or, for the dimensions,

$$[\,x\,] = [\,F^n\,]\,[\,R^s\,]\,[\,E^t\,]. \tag{2}$$

The dimension of the force F is

$$[\,F\,] = [\,ma\,] = MLT^{-2}.$$

Substituting the dimensions of the quantities x, F, R, E into (2), we obtain

$$L = (MLT^{-2})^n\, L^s\, (ML^{-1}T^{-2})^t,$$

or

$$L^1 M^0 T^0 = L^{n+s-t}\, M^{n+t}\, T^{-2n-2t}.$$

We equate the exponents of identical symbols on the left- and right-hand sides of this relation:

$$\begin{cases} 1 = n + s - t, \\ 0 = n + t, \\ 0 = -2n - 2t. \end{cases}$$

Hence

$$t = -n, \qquad s = 1 - 2n.$$

Therefore,

$$x = A_3 F^n R^{1-2n} E^{-n} = A_3 R\left(\frac{F}{ER^2}\right)^n,$$

or

$$\frac{x}{R} = A_3 \left(\frac{F}{ER^2}\right)^n. \tag{1}$$

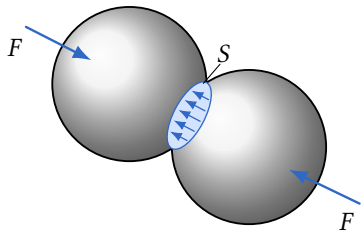


Fig. 10.1. Two identical spheres compressed by opposite forces F; the contact region has cross-section S.

In the general case, external action on the spheres changes the state of the material of the spheres. This change in the state of a material is characterized by the concept of *stress*. The stress of a material in the cross-section S (Fig. 10.1) under the action of external forces F is the quantity

$$\sigma = \frac{F}{S},$$

where S is the area of the cross-section. Stress that arises under the action of forces directed perpendicular to the section considered is called *normal stress* or *pressure*. Stress characterizes the specific load on a material (the load per unit area of the body's cross-section). As the specific loads increase, the material first deforms plastically, and then it fails. Therefore, determining stresses is of practical interest.

II

PART II

Problems 11–20

11 Statically Indeterminate Systems
12 Thin Hoop Rolling Inside a Cylinder
13 Mean Free Path of Air Molecules
14 Heat Capacity of a Gas with a Spring-Loaded Piston
15 Water Vapor in an Evacuated Vessel
16 Gas and Mercury Column in a Vertical Tube
17 Adiabatic Flow of Moist Air over a Mountain Ridge
18 Temperature vs. Altitude in an “Adiabatic” Atmosphere
19 Charging an Accumulator with Motors Used as Generators
20 Cycle Efficiency

PROBLEM 11

Statically Indeterminate Systems

From Kvant, *1978, No. 12*

Veronika's art
p11.png

INTRODUCTORY REMARKS.

Let us consider cases in which problems of statics cannot be solved at all if the bodies of the system are regarded as absolutely rigid. Fig. 11.1 shows an example of such a system. In the case of a rod fixed at both ends (Fig. 11.1), to determine the two unknown reaction forces $\vec{R}_A$ and $\vec{R}_B$ one can write only a single equilibrium condition (in projection onto the X axis).

Such systems are customarily called *statically indeterminate systems*. How then, in these cases, can all the unknown reaction forces be found? It turns out that one can compose additional equations based on the conditions of *compatibility of deformations*. We now show how such problems are solved.

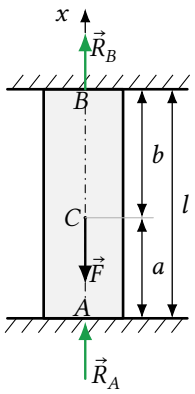


Fig. 11.1

Problem. At point C, which lies on the axis of a cylindrical rod fixed at both ends (Fig. 11.1), an axial force $\vec{F}$ is applied. Determine the reaction forces of the supports $\vec{R}_A$ and $\vec{R}_B$.

Solution. Write the equilibrium condition for the rod in projection onto the X axis:

$$R_A + R_B + F = 0.$$

To determine the two unknowns (R_A and R_B) this single equation is insufficient. We compose one more equation—the equation of compatibility of deformations (displacements).

To this end, mentally release the rod from its support, for example, at the lower section A, and replace the action of the support with the external force $\vec{R}_A$. Denote by Δl_1 the elongation of the rod (displacement of section A) under the action of the force $\vec{F}$, and by Δl_2 the elongation of the rod (again, the displacement of section A) under the action of the force $\vec{R}_A$. In reality the rod does not elongate (section A does not move). Hence the compatibility equation (of deformations/displacements) can be written as

$$\Delta l_1 + \Delta l_2 = 0.$$

The values Δl_1 and Δl_2 are found from Hooke's law, taking into account that

$$N_1 = -F = |\vec{F}| \quad \text{and} \quad N_2 = -R_A :$$

$$\Delta l_1 = \frac{|\vec{F}|\, b}{ES}, \qquad \Delta l_2 = -\frac{R_A\,(a+b)}{ES}.$$

(The force F stretches the portion of the cylinder of length b downward, while the force R_A compresses the entire cylinder of length $a + b$.)

Taking into account that $a + b = l$, we obtain

$$\frac{|F|\, b}{ES} - \frac{R_A\, l}{ES} = 0,$$

whence

$$R_A = |\vec{F}|\, \frac{b}{l}.$$

From the equilibrium condition of the rod we find R_B:

$$R_B = |\vec{F}| - R_A = |\vec{F}|\, \frac{a}{l}.$$

The projections of R_A and R_B are positive; therefore the actual directions of the reaction forces coincide with those adopted in Fig. 11.1.

PROBLEM 12

Thin Hoop Rolling Inside a Cylinder

From Kvant, *1981, No. 3*

Veronika's art
p12.png

Problem. A thin hoop of radius r rolls without slipping along the inner surface of a cylinder of radius R, performing small oscillations about the position of equilibrium (Fig. 12.1). Find the period of these oscillations.

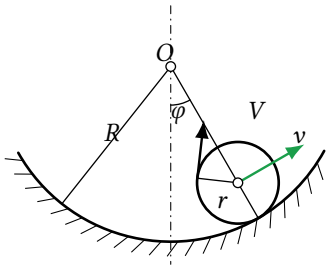


Fig. 12.1

Solution. In most problems of this type, it is very helpful to draw a parallel between the standard equation of simple harmonic motion (for example, a mass oscillating on a spring) and the system described in the problem. To find the angular frequency ω of an oscillatory system (or its period $T = \frac{2\pi}{\omega}$), it is sufficient to write an equation of motion of the form:

$$\ddot{x} + \omega^2 x = 0$$

. In this equation, the role of the displacement x may be played by another physical quantity. Here we choose the angle φ, defined as the angle between the vertical and the straight line joining the hoop's center of mass to the point O.

To compose the equation of motion of the hoop's center of mass we use energy considerations. At an arbitrary moment t the hoop has the kinetic energy E_{k1} associated with the motion of the center of mass along a circle of radius $R-r$; the kinetic energy E_{k2} of rotation about the center of mass; and the potential energy E_p in the Earth's gravitational field. Let the speed of the center of mass of the hoop be v; then the kinetic energy $E_{k1} = mv^2/2$ (m is the mass of the hoop). Since the hoop is thin, i.e. all of its mass is concentrated in a thin ring, the kinetic energy $E_{k2} = mV^2/2$, where V is the linear speed of the rotational motion

of the hoop. For rolling without slipping $V = v$[1], therefore

$$E_{k2} = \frac{mv^2}{2} = E_{k1}.$$

Sketch to be added
ВСТАВИТЬ РИСУНОК К ПОИСКУ ПОТЕНЦ ЭНЕРГИИ

Fig. 12.2

We choose as the zero level of potential energy the level of the hoop's center of mass in the position of stable equilibrium; then the potential energy of the hoop at the given moment is

$$E_p = mg\big((R-r) - (R-r)\cos\varphi\big) = mg(R-r)\big(1-\cos\varphi\big) = \frac{mg(R-r)\,\varphi^2}{2}$$

(the last approximate equality[2] is written taking into account the smallness of the angle φ: $1-\cos\varphi = 2\sin^2(\varphi/2) \approx \varphi^2/2$).

where

$$\Delta h = (R-r) - (R-r)\cos\varphi = (R-r)(1-\cos\varphi)$$

the difference in the center-of-mass height between the equilibrium position and an arbitrary position at angle φ

The total energy of the hoop

$$E_{k1} + E_{k2} + E_p = \frac{mv^2}{2} + \frac{mv^2}{2} + \frac{mg(R-r)\,\varphi^2}{2} = mv^2 + \frac{mg(R-r)\,\varphi^2}{2}$$

remains constant in time. Taking into account the relation between the speed v and the angular speed of the motion of the center of mass along the circle of radius $R-r$:

$v = (R-r)\,\omega = (R-r)\,\varphi'$ (the angular speed is the first time derivative of the angle),

we finally obtain

$$m(R-r)^2(\varphi')^2 + \frac{mg(R-r)\,\varphi^2}{2} = \text{const}.$$

Differentiate both sides of this expression with respect to t:

$$2m(R-r)^2\varphi'\varphi'' + mg(R-r)\varphi\varphi' = 0,$$

or

$$\varphi'\left(\varphi'' + \frac{g}{2(R-r)}\,\varphi\right) = 0.$$

Since the angular speed φ' is not equal to zero, it follows that

$$\varphi'' + \frac{g}{2(R-r)}\,\varphi = 0.$$

This is the equation of motion of the hoop. Consequently, the circular frequency is

$$\omega = \sqrt{\frac{g}{2(R-r)}},$$

and the period of oscillations is

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{2(R-r)}{g}}.$$

NOTES

[1] From the figure, one can see that by summing all the small parts of the hoop and finding

Sketch to be added
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пояснением про качение обруча

Fig. 12.3

the average speed, we obtain

$$V = \frac{2v + 0}{2} = v,$$

where $2v$ is the speed of the top point, and the speed of the bottom point is 0.

[2]

$$\frac{1-\cos\varphi}{2} = \sin^2\left(\frac{\varphi}{2}\right) \quad \text{— half-angle formula}$$

Then,

$$1 - \cos\varphi = 2\sin^2\left(\frac{\varphi}{2}\right)$$

In the small-angle approximation ($\sin\varphi \approx \tan\varphi \approx \varphi$)

$$1 - \cos\varphi = 2\frac{\varphi^2}{4} = \varphi^2/2$$

PROBLEM 13

Mean Free Path of Air Molecules

From Kvant, *1983, No. 2*

Veronika's art
p13.png

Problem. Estimate the mean free path of air molecules under normal conditions. Take the molecular diameter to be $d = 3.7 \cdot 10^{-10}$ m.

Solution. In a volume of air, molecules collide with a large number of other molecules, and in this problem an averaging approach is used: we take a limited region of space—here, a cylinder for convenience of calculation—and assume that a chosen molecule will collide with all molecules inside this cylinder while traveling a distance equal to the cylinder's length. The cylinder's cross-sectional area is taken to be equal to the cross section of the molecule itself.

Sketch to be added
ДОБАВИТЬ РИСУНОК

Fig. 13.1

A molecule collides with another molecule if the distance between their centers becomes smaller than the molecular diameter d. Let in some time interval a molecule travel a path of length l. During this motion it collides with molecules whose centers lie within a broken cylinder (with kinks at the collision points) of length l and cross section πd^2. The number of molecules in this volume is $n\,l\,\pi d^2$ (n is the number of molecules per unit volume) and equals the total number of collisions. We count the path from one collision to the next, that is, the mean free path of the molecule:

$$\lambda = \frac{l}{n\,l\,\pi d^2} = \frac{1}{n\,\pi d^2} = \frac{RT}{N_A p\,\pi d^2} \approx 8.75 \cdot 10^{-8}\ \text{m}.$$

(Here the pressure is $p = 10^5$ Pa, the temperature is $T = 273$ K, and Avogadro's number is $N_A = 6.02 \cdot 10^{23}\ \text{mol}^{-1}$.) The experimental value for such conditions is $6.20 \cdot 10^{-8}$ m. The discrepancy is mainly due to the fact that we treated all molecules, except for the selected one, as stationary. A more detailed analysis (beyond the scope of a school course) shows that accounting for the relative motion of molecules changes the mean free path by a factor of

$1/\sqrt{2}$. Multiplying our result by this factor for the many-particle case, we obtain $6.19 \cdot 10^{-8}$ m.

Now, another aspect is of interest. Under normal conditions the mean free path is about 200 times larger than the molecular diameter, and the ideal-gas model works quite well. But if the pressure increases by a factor of 100–200 (with temperature unchanged), the mean free path becomes comparable with the molecular size. This means that at such pressures—or, more precisely, at such densities that arise at normal temperature and high pressures—molecules are practically never far from one another and their mutual interaction can no longer be neglected. In this case the ideal-gas model becomes inapplicable. (Indeed, the densities of condensed phases—liquids and solids—are roughly 1000 times greater than gas densities; and a gas that is only 5–10 times less dense than a liquid is already not an ideal gas.)

PROBLEM 14

Heat Capacity of a Gas with a Spring-Loaded Piston

From Kvant, *1984, No. 4*

Veronika's art
p14.png

Problem. One mole of an ideal gas is placed in a cylinder with a movable weightless piston fastened to the bottom by a spring whose extension obeys Hooke's law (Fig. 14.1). Determine the heat capacity of the gas, assuming the external pressure is zero and neglecting the length of the unextended spring.

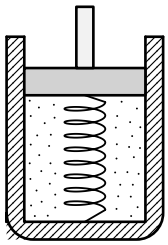


Fig. 14.1

Solution. Let p be the gas pressure, $V = Sx$ the volume occupied by the gas (where S is the cross-sectional area of the cylinder and x is the length of the stretched spring). Denote the spring stiffness by k and write the piston equilibrium condition:

$$pS = kx, \quad \text{or} \quad pS^2 = kV.$$

From the last formula it follows that the changes in pressure Δp and volume ΔV are related by

$$\Delta p = \frac{k}{S^2}\Delta V = \frac{p}{V}\,\Delta V.$$

From the equation of state $pV = \nu RT$, valid for any p, V, T, we obtain

$$(p + \Delta p)(V + \Delta V) = \nu R\,(T + \Delta T).$$

Expanding the last equality and neglecting the small term $\Delta p\,\Delta V$, we get

$$p\,\Delta V + V\,\Delta p = \nu R\,\Delta T,$$

or, taking into account the relation between Δp and ΔV,

$$p\Delta V + \frac{p}{V}V\Delta V = \nu R\Delta T$$

$$2p\Delta V = \nu R\Delta T$$

$$\frac{p}{\nu}\frac{\Delta V}{\Delta T} = \frac{R}{2}.$$

Thus, the heat capacity of the gas is

$$C = C_V + \frac{p}{\nu}\frac{\Delta V}{\Delta T} = C_V + \frac{R}{2}.$$

P.S.

This formula can be obtained as follows:

Let us use the first law of thermodynamics and write out each term explicitly

$$\Delta Q = \Delta A + \Delta U$$

$$\Delta Q = C\nu\Delta T$$
$$\Delta A = p\Delta V$$
$$\Delta U = C_\nu\nu\Delta T$$

Then,

$$\nu c\Delta T = \nu C_\nu\Delta T + p\Delta V$$

$$\boxed{C = C_\nu + \frac{p}{\nu}\frac{\Delta V}{\Delta T}}$$

PROBLEM 15

Water Vapor in an Evacuated Vessel

From Kvant, *1992, No. 6*



Problem. In an evacuated vessel of volume $V = 5$ l we place $m = 1$ g of water. Find the pressure of the water vapor in the vessel at the temperatures $t_1 = 20^\circ$C and $t_2 = 100^\circ$C. The saturated vapor pressures at these temperatures are $p_{n1} = 2.33$ kPa and $p_{n2} = 100$ kPa.

Solution. Before applying the Clapeyron–Mendeleev (ideal gas) equation, it is necessary to ascertain whether the water vapor will be saturated and, if so, what portion of the water will pass into the gaseous state.

Let us begin with the temperature t_1. Suppose, for a first estimate, that all the liquid has evaporated. Then the vapor density would be

$$\rho = \frac{m}{V} = 0.2 \text{ kg/m}^3.$$

Consulting a table of densities of saturated water vapor at various temperatures, we see that at 20°C $\rho_{n1} \approx 0.017$ kg/m^3—our value is much larger. This means that not all of the liquid will evaporate and the vapor above it will be saturated. Therefore, the vapor pressure in the vessel will be

$$p_1 = p_{n1} = 2.33 \text{ kPa}.$$

Direct use of the formula would have led to the result

$$p_1 = \frac{mRT_1}{MV} \approx 27 \text{ kPa (!)},$$

which is more than 10 times greater than the saturated vapor pressure at the given temperature.

Obviously, upon further heating the evaporation of the liquid continues. When it has completely evaporated, the density of the resulting vapor becomes equal to the previously computed value $\rho = 0.2$ kg/m^3. Looking again in the density table, we see that this occurs at the temperature $t_3 = 70^\circ$C and at a pressure equal to the saturated vapor pressure $p_{n3} = 31.7$ kPa. That is, when the temperature reaches 70°C, all the water will evaporate, and the vessel will contain water vapor with a constant ν, which follows the Clapeyron–Mendeleev law (in this case, the volume V is also constant). Then, for isochoric heating from $t_3 = 70^\circ$C to $t_2 = 100^\circ$C, the pressure increase occurs according to the relation:

$$p_2 = \frac{p_{n3} T_2}{T_3} \approx 34.5 \text{ kPa}.$$

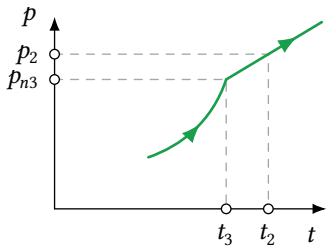


Fig. 15.1

Another way.

The graph shows two segments. The first, from temperature t_1 to t_3, is curved as water evaporates and the relationship between pressure and temperature is nonlinear. The second, between t_3 and t_2, has a constant mass of vapor, so the relationship between pressure and temperature is linear.

Since the density table shows that all the water evaporates below 100°C, at 100°C the vapor is already unsaturated (all the water evaporates at 70°C); its pressure can then be computed more simply—from the Clapeyron–Mendeleev equation:

$$p_2 = \frac{\rho R T_2}{M} \approx 34.5 \text{ kPa}.$$

However, applying this formula to the first state with the temperature $t_1 = 20$°C without the preliminary estimate indicating that the vapor is saturated would have led to an absurd result.

Temperature, °C	Vapor pressure, Pa	Saturated vapor density, g/m^3
−50	4.0	0.039
−20	1.04×10^2	0.89
−10	2.60×10^2	2.36
0	6.10×10^2	4.84
5	8.68×10^2	6.80
10	1.19×10^3	9.40
15	1.69×10^3	12.8
20	2.33×10^3	17.2
25	3.17×10^3	23.0
30	4.24×10^3	30.4
37	6.31×10^3	44.0
40	7.34×10^3	51.1
50	1.23×10^4	82.4
60	1.99×10^4	130
70	3.12×10^4	197
80	4.73×10^4	294
90	7.01×10^4	418
95	8.59×10^4	505
100	1.01×10^5	**598**
120	1.99×10^5	1095
150	4.76×10^5	2430
200	1.55×10^6	7090
220	2.32×10^6	10 200

Fig. 15.2. Table of the density of saturated water vapor at different temperatures

PROBLEM 16

Gas and Mercury Column in a Vertical Tube

From Kvant, *2001, No. 2*



Problem. The lower end of a vertical narrow tube of length $2L$ is sealed, and the upper end is open to the atmosphere (Fig. 16.1a). In the lower half of the tube there is a gas at temperature T_0, and the upper half is filled with mercury. To what *minimum* temperature must the gas in the tube be heated so that it expels all the mercury? The external pressure, expressed in millimeters of a mercury column, is equal to L. *(Answer:* $9T_0/8$.*)*

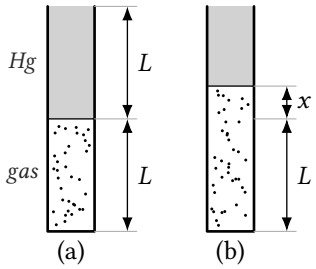


Fig. 16.1

Solution. Many students, even experienced Olympiad contestants, often confuse two similar but physically different concepts. Try to answer this question: how are the ideas of "heating a body" and "supplying it with an amount of heat" related? We will give the answer at the end of the problem, using a short excerpt from the beginning of the article from which this problem is taken.

Select some volume of gas of height x (Fig. 16.1b). It is under the pressure of a mercury column of height $L-x$ and the external pressure $p_0 = \rho g L$, where ρ is the density of mercury. According to Pascal's law, the gas pressure is

$$p = \rho g(2L - x).$$

Let the internal cross-sectional area of the tube be S; then the gas volume is $V = S(L + x)$,

whence

$$x = \frac{V}{S} - L.$$

Taking into account that $SL = V_0$ is the initial gas volume, we write

$$x = \frac{VL}{V_0} - L.$$

Then

$$p = \left(-\frac{p_0}{V_0}\right)V + 3p_0,$$

This is a linear equation of the form $p = aV + b$.

Using the equation of state $pV = \nu RT$, we express the gas temperature in terms of the volume:

$$T = \frac{pV}{\nu R} = \frac{aV^2 + bV}{\nu R} \qquad \text{We obtain a parabola.}$$

To check where the temperature reaches a maximum, we can either take the derivative or use the fact that the dependence is quadratic:

$$V_1 = -\frac{b}{2a}$$

$$T_{max} = \frac{\frac{b^2}{4a} + b\frac{-b}{2a}}{\nu R} = -\frac{b^2}{4a\nu R}$$

Substituting the values of a and b:

$$a = -p_0/V_0 \qquad b = 3p_0$$

We obtain

$$T_{max} = \frac{9p_0V_0}{4\nu R}, \qquad V_1 = \frac{3V_0}{2}$$

Expressing ν from the equation $2p_0V_0 = \nu RT_0$ for the initial state of the gas, we have

$$T = \frac{9T_0}{8}.$$

It would seem that this agrees with the answer quoted. However, analysis of the situation shows that achieving a certain temperature by the system in some processes does not guarantee that the desired process will actually occur—in our case, that the mercury will be expelled. The principal condition for expelling the mercury from the tube is to supply the gas with an amount of heat

$$Q = \frac{49p_0V_0}{32}$$

(We leave the proof of this fact to you), and not merely heating it to the temperature T. If, at

the moment the maximum temperature is reached, the heat supply is stopped, the gas will not be able to perform the work of expansion only at the expense of a decrease of internal energy (we leave it to the reader to verify this independently).

Conclusion. The problem as posed is incorrect. One should require either the determination of the *maximum temperature* reached by the gas in this process (which is what the quoted answer gave), or the determination of the *amount of heat* necessary to expel the mercury from the tube, specifying, for example, that the gas in the tube is monatomic and that the inner cross-sectional area of the tube is S. In this case, taking into account the expressions for p_0 and V_0, we obtain

$$Q = \frac{49\,\rho g L^2 S}{32}.$$

Answer to the question

Heating is an increase in a body's temperature. Temperature T is a state parameter of a system in thermodynamic equilibrium, and internal energy U is a state function. This means that in a given state the system has a definite temperature and internal energy, which do not depend on how the system was brought to that state. This is an experimental fact.

A system changes from one state to another as a result of heat exchange or the performance of mechanical work. Therefore, the amount of heat Q, like the work A done on the system (or the work A' done by the system on external bodies), is related not to the internal energy of the system itself, but to its change. According to the first law of thermodynamics,

$$\Delta U = Q + A \quad \text{or} \quad Q = \Delta U + A'.$$

Thus, supplying a body with some amount of heat does not necessarily lead to its heating.

PROBLEM 17

Adiabatic Flow of Moist Air over a Mountain Ridge

From Kvant, *1996, No. 5*

Veronika's art
p17.png

Problem. A stream of moist air passes adiabatically over a mountain ridge (see figure). At the meteorological stations M_0 and M_3 the atmospheric pressure is the same and equals 100 kPa, while at M_2 it is 70 kPa. The air temperature at station M_0 is 20°C. When the air ascends the slope to the level of station M_1, where the pressure is 84.5 kPa, clouds appear and rain begins. A column of moist air of mass 2000 kg/m^2 climbs from the foot to the ridge in 1500 s. During this time, 2.45 g of rain falls from each kilogram of air.

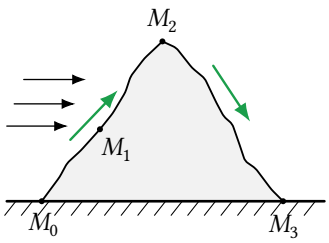


Fig. 17.1

1. Find the air temperature at station M_1, where clouds first appear.
2. Assuming that air density decreases linearly with height, find the altitude of station M_1.
3. What is the temperature at the mountain ridge?
4. How many millimeters of precipitation fall on the slope in three hours if between M_1 and M_2 the rain has constant intensity?
5. What is the air temperature at station M_3? Compare the properties of the air at M_0 and M_3.

Notes. Air may be treated as an ideal gas. The influence of water vapor on the air density may be neglected. The air density at station M_0 is $\rho_0 = 1.189$ kg/m^3. The specific latent heat of vaporization of water in the cloud is $r = 2500$ kJ/kg.

Solution. 1. Since the air rises adiabatically,

$$pV^\gamma = \text{const}.$$

In addition,

$$\frac{pV}{T} = \text{const.}$$

Hence

$$\frac{T_1}{T_0} = \left(\frac{p_1}{p_0}\right)^{1-1/\gamma}, \qquad T_1 = 279.4\ \text{K} = 6.4^\circ\text{C}.$$

2. Find the pressure difference between M_1 and M_0. It is clearly produced by the weight of the "additional" air. Consider an imaginary cylinder of area S and height h_1.

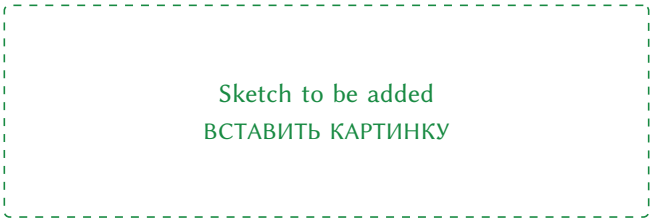


Fig. 17.2

Then

$$-p_1S + p_0S = mg,$$

where m is the mass of the air in the cylinder. Since, by assumption, the air density varies linearly with height, we may write

$$m = h_1S\,\frac{\rho_0 + \rho_1}{2},$$

where ρ denotes the air density. The value ρ_1 is found from the ideal-gas law $p = \frac{\rho}{M}RT$:

$$\frac{p_0}{\rho_0 T_0} = \frac{p_1}{\rho_1 T_1},$$

whence

$$\rho_1 = 1.054\ \text{kg/m}^3.$$

Therefore,

$$h_1 = \frac{p_0 - p_1}{g\,(\rho_0 + \rho_1)/2} = 1400\ \text{m}.$$

3. To compute the temperature at the ridge, we replace the real process by two successive ones and assume that first the air rises adiabatically to the ridge, and then, at the summit, all the rain immediately falls at constant pressure. Then

$$T_2 = T_x + \Delta T.$$

The air temperature just before the rain at the summit is

$$T_x = T_1\left(\frac{p_2}{p_1}\right)^{1-1/\gamma} = 264.8\ \text{K}.$$

After the rain, the amount of heat released in each kilogram of air is

$$Q = rm \quad (\text{here } m = 2.45 \text{ g/kg}),$$

the heating process is isobaric, and the specific heat of air is

$$c_p = \frac{3.5R}{M} = 1000 \text{ J/(kg K)}, \qquad M = 29 \text{ g/mol} - \text{molar mass of air}$$

Hence

$$\Delta T = \frac{Q}{c_p} = \frac{rm}{c_p} = 6.1 \text{ K}.$$

In total we obtain

$$T_2 = T_x + \Delta T = 270.9 \text{ K}.$$

4. From the column of air moving up the slope the precipitation released per second is

$$\frac{2000 \text{ kg/m}^2}{1500 \text{ s}} \cdot \frac{2.45 \text{ g}}{\text{kg}} = 3.3 \cdot 10^{-3} \frac{\text{kg}}{\text{m}^2 \cdot \text{s}},$$

or, since the density of water is 1000 kg/m^3,

$$\frac{3.3 \cdot 10^{-3} \text{ kg/(m}^2 \cdot \text{s)}}{10^3 \text{ kg/m}^3} = 3.3 \cdot 10^{-6} \text{ m/s}.$$

Over three hours ($3 \cdot 3600$ s) the precipitation equals

$$35.3 \cdot 10^{-3} \text{ m} = 35.3 \text{ mm}.$$

5. The air descending beyond the ridge warms adiabatically; therefore

$$T_3 = T_2 \left(\frac{p_3}{p_2} \right)^{1-1/\gamma} = 300 \text{ K} = 27^\circ\text{C}.$$

If there had been no rain, T_3 would have equaled $T_0 = 20^\circ$C. Because of the rain, the air arriving at station M_3 is warmer and drier than at station M_0.

PROBLEM 18

Temperature vs. Altitude in an "Adiabatic" Atmosphere

From Kvant, *1993, No. 5*

Veronika's art
p18.png

Problem. Find the dependence of temperature on height in the so-called adiabatic atmosphere, when the pressure and temperature of an air parcel taken at the Earth's surface are related to the pressure and temperature at height h by the adiabatic equation.

Solution. At height h the pressure $p(h)$, air density $\rho(h)$, and temperature $T(h)$ are linked by the equation of state

$$p(h) = \rho(h)\,\frac{R}{M}\,T(h).$$

The adiabatic equation in the variables p and T has the form

$$p(h)\,T(h)^{\frac{\gamma}{1-\gamma}} = \text{const}.$$

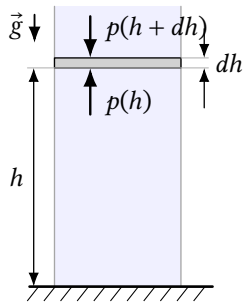


Fig. 18.1

Consider the condition of mechanical equilibrium for a layer of air of unit cross-section and thickness dh with density $\rho(h)$ located at height h. It is clear that the difference between the pressures on the upper and lower faces of the layer is balanced by its weight:

$$p(h+dh) - p(h) = dp = -\,\rho(h)\,g\,dh.$$

(The minus sign indicates that pressure decreases with height; the free-fall acceleration g may be treated as height-independent since the atmospheric thickness is small compared with the Earth's radius.)

We now relate the change in pressure dp to the change in temperature dT within the

layer dh by differentiating the adiabatic relation with respect to temperature (or pressure):

$$dp = \frac{\gamma}{1-\gamma}\frac{p}{T}\,dT.$$

Substituting this expression together with the equation of state into the mechanical-equilibrium condition yields

$$\frac{p}{T} = \rho(h)\frac{R}{M} = \frac{1-\gamma}{\gamma}\frac{dp}{dT}$$

$$\frac{dp}{g\,dh}\frac{R}{M} = \frac{1-\gamma}{\gamma}\frac{dp}{dT}$$

$$dT = -\frac{\gamma-1}{\gamma}\frac{Mg}{R}\,dh.$$

This formula shows that the temperature gradient in an adiabatic atmosphere is constant and equal to

$$\frac{dT}{dh} = -\frac{\gamma-1}{\gamma}\frac{Mg}{R} \approx -10\cdot 10^{-3}\ \text{K/m}.$$

Thus, in an "adiabatic" atmosphere the temperature should decrease by about 10° per kilometer of height, so that the entire atmosphere should be roughly 30 km thick.

Note that there exists a more familiar model—the so-called isothermal atmosphere—in which the dependence of pressure on height is described by the barometric formula

$$p(h) = p(0)\,e^{-h/7.5},$$

where h is measured upward from the Earth's surface in kilometers. Try to compare these models on your own, although, of course, the real atmosphere fits neither one perfectly.

PROBLEM 19

Charging an Accumulator with Motors Used as Generators



From the All-Russian Olympiad, regional stage, 2025

Problem. Sometimes an electric motor can be used as a voltage generator. Consider, as an example, a motor in which the magnetic field is created by a permanent-magnet stator; when the rotor is driven in this field an induced emf appears in the rotor winding. We keep the rotor turning by the tension of a light inextensible cable attached to a massive weight that descends smoothly (practically uniformly) from a certain height H. Note that in this regime the current in the rotor winding is directly proportional to the cable tension. The induced emf is used to charge an accumulator; its charge increases by $Q_1 = 300\ \text{mA}\cdot\text{h}$.

If the same accumulator is connected to two identical motors wired in parallel and the same weight is dropped from the same height on two equally loaded cables, each spinning the rotor of one motor, then the acquired charge becomes $Q_2 = 400\ \text{mA}\cdot\text{h}$.

1. By what factor do the charging times differ in the first and in the second case?
2. By what factor does the charging current of the accumulator in the first and in the second case differ from the current through the rotor circuit of a single motor with its rotor blocked when it is connected to this accumulator?
3. What charge will the accumulator acquire if three identical motors are used in the same manner to charge it?

Assumptions. Variations of the accumulator emf during charging, the internal resistance of the accumulator, and air resistance on the falling weight may be neglected.

Solution. (1) Time ratio. With one generator the cable tension (it equals the weight of the load) is constant ⇒ the rotor current I is constant, hence

$$Q_1 \approx I\,t.$$

With two generators the load is shared equally, i.e. the tension force in each rope is equal to half of the tension for a single cable. The magnetic force on each rotor is proportional to its current, so each generator carries $I' = I/2$. The total charging current remains $I' + I' = I$, hence

$$Q_2 \approx I\,t' \quad \Rightarrow \quad \frac{t'}{t} \approx \frac{Q_2}{Q_1} = \frac{400}{300} = \frac{4}{3}.$$

(2) Current comparison with a locked rotor. During the descent the weight's potential

energy goes into the work of charging the accumulator and into Joule heat in the rotor circuit.

Let ε be the constant (according to the problem statement) emf of the accumulator, and R the resistance of the rotor circuit. The work done on the accumulator is $A = \varepsilon Q_1$; the rate of heat generation is $P = I^2 R$.

If the mass of the load is m, then the law of conservation of energy for charging with one generator has the form

$$mgH \approx \varepsilon Q_1 + I^2 R t = Q_1\left(\varepsilon + \frac{RQ_1}{t}\right). \tag{1}$$

For two generators

$$mgH \approx \varepsilon Q_2 + 2(I')^2 R t' = \varepsilon Q_2 + \frac{1}{2} I^2 R t' = Q_2\left(\varepsilon + \frac{RQ_1}{2t}\right), \tag{2}$$

where we used $Q_1/t \approx Q_2/t'$.

Let us compare the two expressions for mgH

$$Q_2 = Q_1 \cdot \frac{2(\varepsilon t + RQ_1)}{2\varepsilon t + RQ_1} = Q_1 \cdot \frac{2x+2}{2x+1}, \qquad x \equiv \frac{\varepsilon t}{RQ_1}. \tag{3}$$

Note that x equals the ratio of the current through a stationary (blocked) rotor directly connected to the accumulator, $I_0 = \varepsilon/R$, to the charging current I: $x = I_0/I$.

From (3) we find

$$\frac{I_0}{I} = x = \frac{2Q_1 - Q_2}{2Q_2 - 2Q_1} = \frac{2 \cdot 300 - 400}{2 \cdot 400 - 2 \cdot 300} = 1.$$

Thus, in both the first and second cases the charging current equals the current I_0 that would flow through a locked rotor connected to the accumulator.

(3) Three generators. Repeating the reasoning for three generators one obtains

$$Q_3 = Q_1 \cdot \frac{3x+3}{3x+1} = \frac{3Q_1 Q_2}{4Q_1 - Q_2} = \frac{3 \cdot 300 \cdot 400}{4 \cdot 300 - 400} = 450 \text{ mA} \cdot \text{h}.$$

Answers.

$$\boxed{t'/t = \frac{4}{3}, \qquad I_0/I = 1, \qquad Q_3 = 450 \text{ mA} \cdot \text{h}}$$

PROBLEM 20

Cycle Efficiency

From Kvant, *2011, No. 2*

Veronika's art
p20.png

Problem. Determine the efficiency of the cycle $ABCA$ shown in Fig. 20.1.

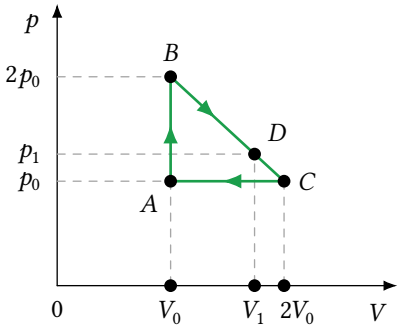


Fig. 20.1

Solution. The work of the gas over the cycle is found immediately:

$$A = \frac{1}{2}\, p_0 V_0 \,.$$

To find the heat Q_1 supplied to the gas we have to work harder. On segment AB heat is unquestionably supplied, and

$$Q_{AB} = U_B - U_A = \frac{3}{2} \cdot 2p_0 V_0 - \frac{3}{2} p_0 V_0 = \frac{3}{2}\, p_0 V_0 \,.$$

On segment BC the pressure continuously decreases, the volume continuously increases, while the temperature behaves more subtly—first it increases up to some point D, and then decreases, returning at point C to its initial value (at point B). This means that heat is supplied to the gas not over the whole segment BC, but only over the stage BD. Find Q_{BD}.

Write the first law of thermodynamics for a small amount of heat:

$$\Delta Q = \Delta U + p\,\Delta V = \Delta\Big(\frac{3}{2} pV\Big) + p\,\Delta V = \frac{5}{2} p\,\Delta V + \frac{3}{2} V\,\Delta p$$

(we used $\Delta(pV) = p\,\Delta V + V\,\Delta p$). Setting ΔQ to zero, we obtain

$$\frac{\Delta p}{\Delta V} = -\frac{5}{3}\,\frac{p}{V} \,.$$

From the slope of the graph we find

$$\frac{\Delta p}{\Delta V} = -\frac{p_0}{V_0}, \quad \text{i.e.} \quad \frac{p_0}{V_0} = \frac{5}{3}\,\frac{p}{V}\,.$$

The equation of the straight line BC is easily written from two known points:

$$p = -\,\frac{p_0}{V_0}\,V + 3p_0\,.$$

Thus, for the coordinates of point D we obtain

$$p_1 = \frac{9}{8}\,p_0, \qquad V_1 = \frac{15}{8}\,V_0$$

(it is clear that p_1V_1 lies between the corresponding products for points B and C). Hence

$$\begin{aligned} Q_{BD} &= \left(U_D - U_B\right) + \frac{1}{2}\,(2p_0 + p_1)\,(V_1 - V_0) \\ &= \frac{3}{2}\,p_1V_1 - \frac{3}{2}\cdot 2p_0V_0 + \frac{1}{2}\cdot\frac{7}{8}\cdot\frac{25}{8}\,p_0V_0 = \frac{49}{32}\,p_0V_0\,. \end{aligned}$$

Therefore,

$$Q_1 = Q_{AB} + Q_{BD} = \frac{3}{2}\,p_0V_0 + \frac{49}{32}\,p_0V_0 = \frac{97}{32}\,p_0V_0\,,$$

and

$$\eta = \frac{A}{Q_1} = \frac{(1/2)p_0V_0}{(97/32)p_0V_0} = \frac{16}{97} \approx 16.5\%.$$

III

PART III

Problems 21–30

21 Piston Motion During Heating and Evaporation of Water
22 Adiabatic Compression of Saturated Water Vapor
23 Charged Particle at the Center of a Charged Ring
24 Surface Charge Densities on Two Parallel Metal Plates
25 Two Thin Concentric Conducting Spheres
26 Potential of a Charged Conducting Sphere in the Presence of a Point Charge
27 Circuit of Capacitors
28 Planar Diode with Space Charge
29 Two Metallic Rings
30 Pseudo-experiment “Viscous Friction”

PROBLEM 21

Piston Motion During Heating and Evaporation of Water

From Kvant, *1974, No. 1*

Veronika's art
p21.png

Problem. In a long cylindrical vessel with a weightless piston of area $S = 0.01$ m^2 there is water of mass $m_w = 0.01$ kg at temperature $T_0 = 273$ K. The piston fits tightly against the vessel walls, moves without friction, and at the initial moment is at the water surface (Fig. 21.1a). The water is heated with a heater of constant power $N = 1$ kW; heat losses through the walls and the piston can be neglected. Draw the graph of the piston height h above the initial level as a function of time t. Atmospheric pressure is $p_0 = 10^5$ N/m^2.

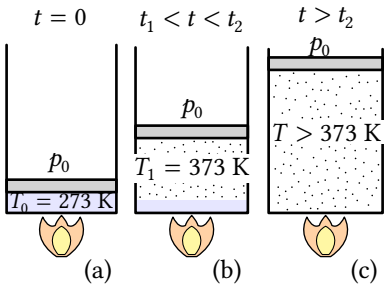


Fig. 21.1

Solution. Until the saturated vapor pressure of water becomes equal to the normal atmospheric pressure (this occurs at temperature $T_1 = 373$°K), the piston stays at the water surface, rising only due to the thermal expansion of the liquid (Fig. 21.1a). The heating time t_1 required to raise the water to temperature T_1 is found easily:

$$t_1 = \frac{c\, m_w (T_1 - T_0)}{N} = \frac{4 \cdot 10^3 \cdot 0.01 \cdot 100}{10^3} = 4 \text{ s}$$

(here $c = 4 \cdot 10^3$ J/(kg · K) is the specific heat of water).

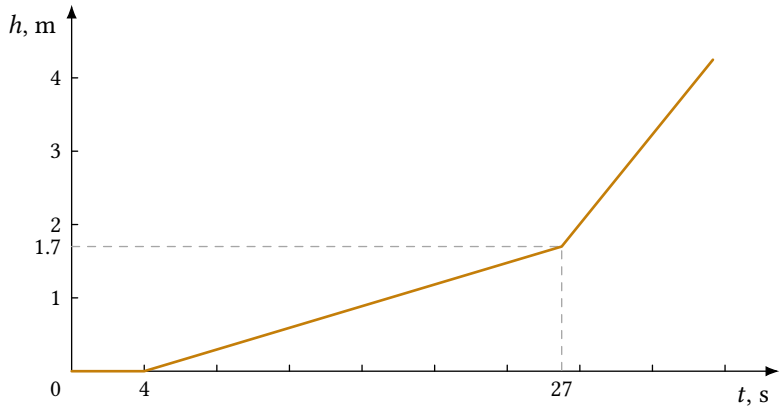


Fig. 21.2

The corresponding part of the graph (see Fig. 21.2) is drawn as a horizontal line:

$$h = 0 \qquad \text{for } 0 < t < t_1,$$

since the coefficient of thermal expansion of water is negligible.

At $T_1 = 373$ K the saturated vapor pressure becomes equal to the external pressure, and the piston begins to rise above the water surface (Fig. 21.1b): saturated water vapor forms beneath the piston. Since the heater power is constant, the vapor mass

$$m = \frac{N\,(t - t_1)}{L} \qquad \big(L = 2.3 \cdot 10^6 \text{ J/kg is the latent heat of vaporization of water}\big)$$

is proportional to the heating time after reaching T_1. From the Clapeyron–Mendeleev equation ($pV = \frac{m}{\mu}RT$) the vapor volume is

$$V = \frac{RT_1\,m}{p_0\mu} = \frac{RT_1 N}{p_0\mu L}\,(t - t_1)$$

(since T_1 and p_0 are constant). Therefore, for the piston height we finally obtain

$$h = \frac{V}{S} = \frac{RT_1 N}{p_0\mu LS}\,(t - t_1) \qquad \text{for } t_1 < t < t_2,$$

where

$$t_2 = t_1 + \frac{m_\text{w} L}{N} = 4 + \frac{0.01 \cdot 2.3 \cdot 10^6}{10^3} \approx 27 \text{ s},$$

the instant at which all the water in the vessel has evaporated. On the graph this section is a straight line with slope α, with

$$\tan\alpha = \frac{RT_1 N}{p_0\mu LS} = \frac{8.3 \cdot 10^3 \cdot 373 \cdot 10^3}{10^5 \cdot 18 \cdot 2 \cdot 10^6 \cdot 0.01} \approx 0.08 \text{ (m/s)}.$$

After this moment the temperature of the vapor increases and it ceases to be saturated. The temperature change of the vapor follows

$$\Delta T = \frac{N\,(t - t_2)}{c_p m},$$

$c_p = 2 \cdot 10^3 \ \dfrac{\text{J}}{\text{kg} \cdot \text{deg}}$ – the specific heat of water vapor for isobaric heating.

The corresponding change of volume is found from the Clapeyron–Mendeleev equation:

$$\Delta V = \frac{mR\Delta T}{\mu p_0} = \frac{NR\,(t - t_2)}{\mu c_p p_0}.$$

For the piston height we have

$$h = h_1 + \frac{NR}{\mu c_p p_0 S}\,(t - t_2) \qquad (t > t_2),$$

where

$$h_1 = \frac{m_\text{w} R T_1}{\mu p_0 S} = \frac{0.01 \cdot 8.3 \cdot 10^3 \cdot 373}{18 \cdot 10^5 \cdot 0.01} \approx 1.7 \text{ m}$$

is the piston height at the end of evaporation. On the graph this section is a straight line with slope β, where

$$\tan\beta = \frac{NR}{\mu c_p p_0 S} = \frac{10^3 \cdot 8.3 \cdot 10^3}{18 \cdot 2 \cdot 10^3 \cdot 10^5 \cdot 0.01} \approx 0.2 \text{ (m/s)}.$$

Remark on boiling. When solving the last part of the problem one may ask: will the water in the vessel boil at $T_1 = 373$ K? As is known, for boiling it is necessary that gas bubbles at the bottom of the vessel begin to grow due to evaporation of liquid into them. Strictly speaking, this occurs when the saturated vapor pressure exceeds the external atmospheric pressure by the sum of the excess (capillary) pressure

$$p_1 = \frac{2\sigma}{r},$$

where σ is the surface tension coefficient and r is the bubble radius, and the hydrostatic pressure

$$p_2 = \rho g h,$$

where ρ is the density and h is the liquid height in the vessel. Therefore, at normal external pressure boiling begins at a temperature slightly higher than 373°K, and in the present problem the liquid will *not* boil. Usually, noticeable superheating of a liquid in a vessel does not occur, since the additional pressure is small compared with the external pressure. To superheat a liquid, it must be carefully degassed.

PROBLEM 22

Adiabatic Compression of Saturated Water Vapor

From Kvant, *1998, No. 3*



Problem. Water vapor is contained in a thermally insulated chamber at temperature $T = 300$ K. There is also liquid water present, whose mass is small compared with the mass of the vapor. In the course of adiabatic compression the temperature of the vapor increases by $\Delta T = 1$ K, and a portion of the water evaporates. Find the relative increase of the vapor mass in the chamber. The latent heat of vaporization at $T = 300$ K is $\lambda = 2370$ kJ/kg. The vapor may be regarded as an ideal gas with molar heat capacity at constant volume

$$C_V = 3R = 25\,\mathrm{J\,(mol \cdot K)}.$$

The heat capacity of the liquid water is negligible. It is also known that small temperature changes $\Delta T/T$ are related to relative changes of the saturated vapor pressure by

$$\frac{\Delta p}{p} = k\,\frac{\Delta T}{T}, \qquad k = 17.$$

Solution. The pressure p, volume V, mass m, and temperature T of saturated water vapor of molar mass M ($M = 18$ g/mol) are connected by the ideal-gas equation of state

$$pV = \frac{m}{M}RT.$$

Small variations of these quantities satisfy

$$p\,\Delta V + V\,\Delta p = \frac{m}{M}R\,\Delta T + \frac{\Delta m}{M}RT.$$

Readers familiar with differentiation will recognize where this relation comes from; others may take it as a known fact for now.

By energy conservation, during adiabatic compression the work of the external forces, equal to $-p\,\Delta V$ (the vapor volume decreases), is spent on evaporating the mass Δm of water and on increasing the internal energy of the vapor by $mC_V\Delta T/M$ (the change of vapor mass is small). Therefore,

$$-p\,\Delta V = \lambda\,\Delta m + \frac{m}{M}C_V\,\Delta T.$$

By the condition of the problem,

$$\frac{\Delta p}{p} = k\,\frac{\Delta T}{T}.$$

From all these relations we obtain the desired quantity:

$$\boxed{\frac{\Delta m}{m} = \frac{kR - R - C_V}{M\lambda + RT}\,\Delta T}$$

and numerically

$$\frac{\Delta m}{m} \approx \frac{13}{18}\,\frac{\Delta T}{T} \approx 0.0024.$$

PROBLEM 23

Charged Particle at the Center of a Charged Ring

From Kvant, *1972, No. 6*

Veronika's art
p23.png

Problem. At the center of a wire ring of radius R there is a charged particle that has speed v_0 (see Fig. 23.1). The ring carries charge $+Q$, and the particle has charge $-q$. What speed v will the particle have far from the ring?

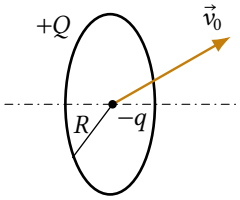


Fig. 23.1

Solution. The electric potential at the center of the ring equals the sum of the potentials created by the individual infinitesimal charges ΔQ_i on the ring. Since each ΔQ_i may be treated as a point charge,

$$\varphi = \sum_i \frac{\Delta Q_i}{R} + c = \frac{Q}{R} + c.$$

where c is some constant, since the potential is a relative quantity, defined only up to an additive constant (very often one takes the potential at infinity to be 0, but this is not actually a "truth"—it is simply a convenient choice of the zero reference level from which the counting starts).

The potential energy of interaction of the charge $-q$ with the charge $+Q$ is

$$\Pi = -q\varphi = -\frac{qQ}{R} - qc.$$

The total energy at the initial moment is therefore

$$E_\text{i} = \frac{mv_0^2}{2} - \frac{qQ}{R} - qc.$$

In the final state (far from the ring) the particle's kinetic energy is $\frac{mv^2}{2}$, while its potential energy is $-qc$, because the distance-dependent term of the potential energy vanishes. By

energy conservation,

$$\frac{mv_0^2}{2} - \frac{qQ}{R} - qc = \frac{mv^2}{2} - qc.$$

Hence

$$v = \sqrt{v_0^2 - \frac{2Qq}{mR}}.$$

As we see, the result does not depend on the value of the constant c.

PROBLEM 24

Surface Charge Densities on Two Parallel Metal Plates

From Kvant, *1973, No. 3*

Veronika's art
p24.png

Problem. Two metallic plates are arranged in parallel (see Fig. 24.1). Plate 1 is given a charge Q; plate 2 is uncharged. The plates have the same area S. Find the surface charge densities on both sides of each plate.

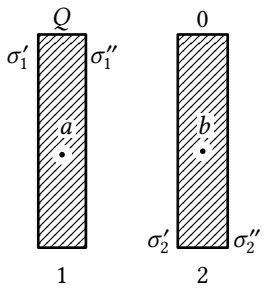


Fig. 24.1

Solution. Denote the surface charge densities on the left and right sides of plate 1 by σ_1' and σ_1'', and on plate 2 by σ_2' and σ_2'', respectively.

Sketch to be added
ВСТАВИТЬ КАРТИНКУ С НАПРАВЛЕНИЯМИ ПОЛЕЙ

Fig. 24.2

We take all densities to be positive; a negative value in the answer means that the charge on that face is negative.

Then we can write

$$(\sigma_1' + \sigma_1'')S = Q, \qquad (\sigma_2' + \sigma_2'')S = 0. \tag{1}$$

According to the superposition principle, the electric field at any point (including inside a plate) equals the sum of the fields produced by the four charged planes with surface densities

σ_1', σ_1'', σ_2', and σ_2''. For the field at point a (inside plate 1; see Fig. 24.1) we have

$$E_a = \frac{\sigma_1'}{2\varepsilon_0} - \frac{\sigma_1''}{2\varepsilon_0} - \frac{\sigma_2'}{2\varepsilon_0} - \frac{\sigma_2''}{2\varepsilon_0}, \tag{2}$$

and at point b (inside plate 2)

$$E_b = \frac{\sigma_1'}{2\varepsilon_0} + \frac{\sigma_1''}{2\varepsilon_0} + \frac{\sigma_2'}{2\varepsilon_0} - \frac{\sigma_2''}{2\varepsilon_0}. \tag{3}$$

The electrostatic field inside conductors is zero, therefore

$$E_a = E_b = 0. \tag{4}$$

Solving equations (1)–(4) jointly (the details are given below), we find

$$\sigma_1' = \sigma_1'' = \frac{Q}{2S}, \qquad \sigma_2' = -\frac{Q}{2S}, \qquad \sigma_2'' = \frac{Q}{2S}.$$

Thus, each face of plate 1 carries a positive surface density $Q/(2S)$. On plate 2, the face turned toward plate 1 has surface density $-Q/(2S)$, while its outer face has $+Q/(2S)$. The net charge on plate 2 is zero, as required.

DETAILS OF THE DERIVATION

1.

$$\sigma_1' - \sigma_1'' - \sigma_2' - \sigma_2'' = \sigma_1' - \sigma_1'' - (\sigma_2' + \sigma_2'') = 0 \Rightarrow \sigma_1' = \sigma_1'' = \frac{Q}{2S}$$

2.

$$\sigma_1' + \sigma_1'' + \sigma_2' - \sigma_2'' = 0$$

Considering that

$$\sigma_2'' = -\sigma_2' \qquad \text{and} \qquad \frac{Q}{S} = \sigma_1' + \sigma_1'',$$

we obtain

$$\frac{Q}{S} = -2\sigma_2'$$

$$\sigma_2' = -\frac{Q}{2S}$$

$$\sigma_2'' = \frac{Q}{2S}$$

PROBLEM 25

Two Thin Concentric Conducting Spheres

From Kvant, *1999, No. 4*

Veronika's art
p25.png

Problem. Two thin concentric conducting spheres of radii R_1 and R_2 ($R_1 < R_2$) carry charges q_1 and q_2, respectively. Compute the potentials of the spheres and the energy of the system. What charge will remain on the inner sphere if it is grounded[1] (see Fig. 25.1)? How will the energy of the system change in this case?

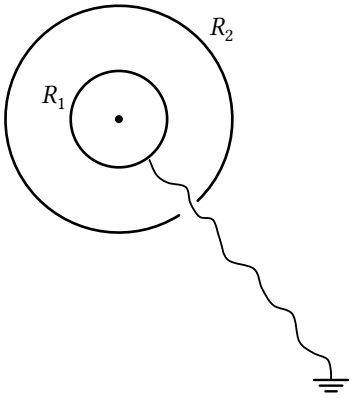


Fig. 25.1

Solution. The potential at any point of space can be found by the principle of superposition—as the sum of the potential $\varphi_1(r)$ created by the charges of the first sphere and the potential $\varphi_2(r)$ created by the second sphere. For every point in the exterior region ($r \geq R_2$) both contributions must be computed using the formula for the potential of a point charge.

$$\varphi = \frac{1}{4\pi\varepsilon_0}\frac{q}{r} \quad \text{when } r \geq R$$

Hence, the potential of the outer sphere ($r = R_2$) is

$$\varphi(R_2) = \frac{1}{4\pi\varepsilon_0}\frac{q_1 + q_2}{R_2}. \tag{1}$$

In the region between the spheres ($R_1 < r < R_2$) the contribution of the inner sphere is

[1] The grounding wire passes through a small hole in the outer sphere without touching it.

given by the point-charge formula, and the contribution of the outer sphere is constant:

$$\varphi(r) = \frac{1}{4\pi\varepsilon_0}\,\frac{q_1}{r} + \frac{1}{4\pi\varepsilon_0}\,\frac{q_2}{R_2}. \tag{2}$$

Setting $r = R_2$ in (2) reproduces (1); setting $r = R_1$ gives the potential of the inner sphere:

$$\varphi(R_1) = \frac{1}{4\pi\varepsilon_0}\,\frac{q_1}{R_1} + \frac{1}{4\pi\varepsilon_0}\,\frac{q_2}{R_2}. \tag{3}$$

The energy of this system of charges is

$$W = \frac{1}{2}\,q_1\varphi(R_1) + \frac{1}{2}\,q_2\varphi(R_2) = \frac{1}{8\pi\varepsilon_0}\left(\frac{q_1^2}{R_1} + \frac{2q_1q_2}{R_2} + \frac{q_2^2}{R_2}\right). \tag{4}$$

The first and third terms are the self-energies of the spheres; the middle term is their interaction energy.

After grounding the inner sphere, its potential becomes zero. Using (3) we obtain the equation for the new charge q_1' on this sphere:

$$0 = \frac{1}{4\pi\varepsilon_0}\,\frac{q_1'}{R_1} + \frac{1}{4\pi\varepsilon_0}\,\frac{q_2}{R_2}, \qquad \Rightarrow \qquad q_1' = -\,q_2\,\frac{R_1}{R_2}.$$

With (1) the new potential of the outer sphere is

$$\varphi'(R_2) = \frac{1}{4\pi\varepsilon_0}\,\frac{q_1' + q_2}{R_2} = \frac{1}{4\pi\varepsilon_0}\,\frac{q_2\,(R_2 - R_1)}{R_2^2}.$$

Since the potential of the inner sphere is now zero, the energy of the system in the final state equals

$$W' = \frac{1}{2}\,q_2\,\varphi'(R_2) = \frac{1}{8\pi\varepsilon_0}\,\frac{q_2^2(R_2 - R_1)}{R_2^2}.$$

We see that the final energy of the system is less than the initial energy. This is natural: the decrease in electrostatic energy equals the amount of heat released during the redistribution of charge.

Remark.

The calculation of the potential of an isolated sphere at a distance $r < R$:

$$\Delta\varphi = \varphi_\infty - \varphi_r = -\int_r^\infty E\,dr = -\left(\int_r^R E\,dr + \int_R^\infty E\,dr\right) = 0 + \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{R} - \frac{1}{\infty}\right) = \frac{q}{4\pi\varepsilon_0 R}$$

$$E = 0 \qquad (r < R)$$

$$E = \frac{q}{4\pi\varepsilon_0 r^2} \qquad (r > R)$$

Here the potential at infinity is taken to be zero, $\varphi_\infty = 0$.

PROBLEM 26

Potential of a Charged Conducting Sphere in the Presence of a Point Charge

From Kvant, *1999, No. 4*

Veronika's art
p26 . png

Problem. A conducting sphere of radius R carries a charge Q. What will be the potential of the sphere if a point charge q is placed at a distance l from its center? Discuss the cases $l > R$ and $l < R$.

Solution. At first glance, the problem seems difficult because the spherical symmetry is broken and the surface-charge distribution on the sphere becomes nonuniform. Indeed, to obtain a complete description, i.e. to determine the charge distribution on the sphere and the field around it, is not simple (though possible). One can do this, for example, by the method of electrostatic images. However, the question posed here can be answered quite simply, relying on the symmetry of the sphere and on the uniqueness theorem.

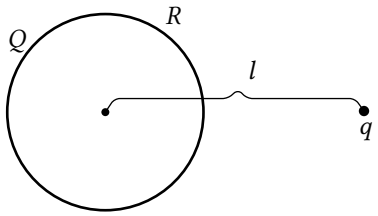


Fig. 26.1

We begin with the case $l > R$ (Fig. 26.1). In this case the potentials of all points on the sphere are equal, and it suffices to find the potential at some one point. Naturally, we choose the center of the sphere. The contribution from the charges distributed over the surface of the sphere is computed by the following formula and equals

$$\sum \frac{1}{4\pi\varepsilon_0}\frac{\Delta Q_i}{R} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{R}$$

(since this computation does not use the uniformity of the charge distribution—the answer depends only on the total charge of the sphere). It remains to take into account the contribution of the point charge and to write

$$\varphi(R) = \varphi(0) = \frac{1}{4\pi\varepsilon_0}\frac{Q}{R} + \frac{1}{4\pi\varepsilon_0}\frac{q}{l}. \tag{1}$$

It is clear that the potential of the sphere, when a point charge is placed near it, changes by the amount of the potential created by this charge at the center of the sphere. To avoid misunderstanding we note that there exists a *unique* distribution of charges over the surface of the sphere for which the potential of *all* interior points of the sphere equals the value obtained.

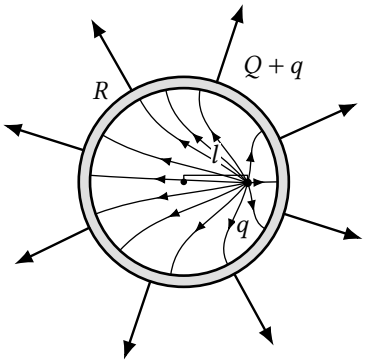


Fig. 26.2

Now turn to the case $l < R$ (Fig. 26.2). Since the charge now lies inside the sphere, the field inside the sphere is nonzero and the potentials of different interior points are not equal to each other. Even so, it is easy to determine the sphere's potential; one must only focus attention not on the interior of the sphere but on the exterior space surrounding it. It turns out that the field in the exterior space does not depend on the position of the charge q inside the sphere, i.e. when the charge is moved within the interior region, the field in the exterior region does not change.

This statement is valid for a hollow conductor of *any* shape and follows from the uniqueness theorem. The field in the exterior empty space is uniquely determined by the following conditions: 1) the potential at infinity is zero; 2) the potential on the surface of the conductor takes some constant value; 3) the total charge inside that surface is known, i.e. the total number of field lines originating on the conductor's surface is fixed. There exists a unique field satisfying these conditions.

For a spherical conductor, the field in the exterior region coincides with the field of a point charge $Q + q$ located at the center. In this case the charge on the sphere is distributed as follows: a charge $-q$ resides on the inner surface of the sphere, because all field lines that originate on the charge q terminate there, and on the outer surface the charge $Q + q$ is distributed uniformly. Therefore, in this case the potential of the sphere equals

$$\varphi(R) = \frac{1}{4\pi\varepsilon_0}\frac{Q}{R} + \frac{1}{4\pi\varepsilon_0}\frac{q}{R}, \tag{2}$$

and it does not depend on the distance l.

Now let us try to answer the following question: what will be the potential of a conducting sphere carrying a charge Q in the presence of two point charges q_1 and q_2 located at distances l_1 and l_2 from the center of the sphere ($l_1 < R < l_2$)? It may seem that neither of the reasonings used above for a single charge can be applied here. Indeed, for the first

part of the problem the important fact was that the field inside the sphere is zero, while for the second—that there is no charge on the sphere. But the uniqueness theorem allows us to answer the question using the superposition of the two cases of the placement of a charge relative to the sphere considered above.

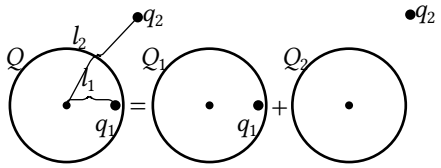


Fig. 26.3

We split the problem into two parts. First, consider the charge q_1 at a distance l_1 from the center of the sphere that carries charge Q_1, and then the charge q_2 at a distance l_2 from the center of the sphere that carries charge Q_2, with $Q_1 + Q_2 = Q$ (Fig. 26.3). The potential of the sphere in the first case is determined by formula (2), and in the second case by formula (1). Now superimpose the first system on the second. Since the potentials of all points of the sphere were constant in each case, upon superposition the potentials will also be constant, and the charge of the sphere will be equal to Q. Consequently, the charge distribution obtained by overlaying the two surface-distribution patterns is the correct one (uniqueness theorem). For the potential of the sphere we obtain

$$\varphi(R) = \frac{1}{4\pi\varepsilon_0}\frac{Q}{R} + \frac{1}{4\pi\varepsilon_0}\frac{q_2}{l_2} + \frac{1}{4\pi\varepsilon_0}\frac{q_1}{R}.$$

This result naturally generalizes to any number of point charges. It is interesting to note that a kind of "equivalence" of point charges and charged spheres follows from here in problems where one needs to determine the potential of a conducting sphere. Since the contribution of a point charge to the potential of the sphere depends only on the distance l between this charge and the center of the sphere, the potential of the sphere will not change if we "smear" this charge over the surface of an imaginary sphere of radius l.

PROBLEM 27

Circuit of Capacitors

From Kvant, *2011*

Veronika's art
p27 . png

Problem. In the network shown in Fig. 27.1 the capacitances are

$$C_1 = C, \quad C_2 = 2C, \quad C_3 = 3C, \quad C_4 = 4C, \quad C_5 = 5C, \quad C_6 = 6C, \quad C_7 = 7C,$$
$$C_8 = 8C, \quad C_9 = 9C, \quad C_{10} = 10C, \quad C_{11} = 11C, \quad C_{12} = 12C, \qquad C = 1\ \mu\text{F}.$$

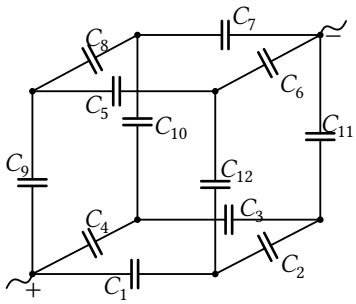


Fig. 27.1

It is required to find the total (equivalent) capacitance of the resulting chain of capacitors.

Solution. When solving problems of this type, one can use the symmetry method, but it cannot be applied in all cases. In such situations, one resorts to a very important property of capacitor circuits: in a series connection of capacitors, the net charge on the conductors between the adjacent plates of two capacitors is zero.

Then, assign charges and node potentials arbitrarily, as shown in Fig. 27.2. Next, identify the regions (nodes) where the net charge is zero; these are the vertices of the cube. We then write the *charge-balance* equations:

$$q_1 + q_4 - q_9 = 0, \qquad q_8 + q_9 - q_5 = 0, \qquad q_2 - q_1 - q_{12} = 0, \tag{1}$$
$$q_5 + q_6 + q_{12} = 0, \qquad q_{10} - q_8 - q_7 = 0, \qquad q_3 - q_4 - q_{10} = 0, \tag{2}$$
$$-q_2 - q_3 - q_{11} = 0, \qquad q_7 - q_6 + q_{11} = 0. \tag{3}$$

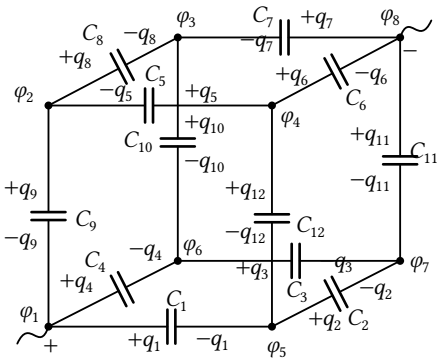


Fig. 27.2

The remaining equations are obtained from the definition of a capacitor, expressing potential differences via charges and capacitances:

$$\varphi_1 - \varphi_5 = \frac{q_1}{C}, \qquad \varphi_5 - \varphi_7 = \frac{q_2}{2C}, \qquad \varphi_6 - \varphi_7 = \frac{q_3}{3C}, \qquad \varphi_1 - \varphi_6 = \frac{q_4}{4C}, \tag{4}$$

$$\varphi_4 - \varphi_2 = \frac{q_5}{5C}, \qquad \varphi_4 - \varphi_8 = \frac{q_6}{6C}, \qquad \varphi_8 - \varphi_3 = \frac{q_7}{7C}, \qquad \varphi_2 - \varphi_3 = \frac{q_8}{8C}, \tag{5}$$

$$\varphi_2 - \varphi_1 = \frac{q_9}{9C}, \qquad \varphi_3 - \varphi_6 = \frac{q_{10}}{10C}, \qquad \varphi_8 - \varphi_7 = \frac{q_{11}}{11C}, \qquad \varphi_4 - \varphi_5 = \frac{q_{12}}{12C}. \tag{6}$$

We must determine the overall capacitance of the system. The charge drawn from the positive terminal flows only onto the capacitors with capacitances C_1, C_4, and C_9. Therefore, it suffices to find the charges on these capacitors. Denoting

$$U = \varphi_1 - \varphi_8,$$

and solving the resulting system, we obtain

$$q_1 \approx 2.04\,CU, \qquad q_4 \approx 11.16\,CU, \qquad q_9 \approx 13.20\,CU.$$

The equivalent capacitance of the chain is then

$$C_{\text{eq}} = \frac{q_1 + q_4 - q_9}{U} \approx 26.4\,C = 26.4\ \mu\text{F}.$$

PROBLEM 28

Planar Diode with Space Charge

From Kvant, *1993, No. 6*

Veronika's art
p28.png

Problem. Because of the presence of a space charge in the interelectrode region of a planar diode, the potential distribution shown in Fig. 28.1 is established. Here the minimal potential is $U_m = -2.25$ V, and the anode potential is $U_a = 33.75$ V. What minimal energy must an electron have at the exit from the cathode in order to reach the anode? What is the electron's time of flight at this energy? The charge-to-mass ratio of the electron is $\gamma = 1.76 \cdot 10^{11}$ C/kg.

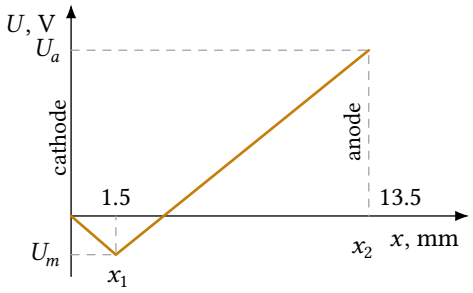


Fig. 28.1

Solution. In solving the problem we must first analyze the distribution of the electric-field strength $E(x)$ between the cathode and the anode. For this purpose we use the relation between the field strength and the potential variation:

$$E = -\Delta U/\Delta x.$$

On the interval $0 \le x \le x_1$ (near the cathode),

$$U(x) = -\frac{|U_m|}{x_1}\, x.$$

Therefore,

$$E(x) = \frac{|U_m|}{x_1},$$

and we have a region of a uniform field whose strength equals

$$E_1 = \frac{|U_m|}{x_1},$$

directed from the cathode to the anode.

Consider now the interval $x_1 < x \le x_2$. Here the potential distribution has the form

$$U(x) = \frac{U_a - U_m}{x_2 - x_1}\, x - \frac{U_a x_1 - U_m x_2}{x_2 - x_1},$$

and the field strength is

$$E(x) = -\frac{U_a - U_m}{x_2 - x_1}.$$

Thus, on this interval we again have a uniform field with strength

$$E_2 = -\frac{U_a - U_m}{x_2 - x_1},$$

but now directed from the anode to the cathode. At $x = x_1$ the function $E(x)$ has a discontinuity; this is a result of our simplified model of $U(x)$—in reality this jump is absent.

We can now describe the electron motion in the interelectrode space of the diode.

On the interval $0 \le x \le x_1$ the electron moves with uniform deceleration, with acceleration

$$a_1 = -\frac{e|U_m|}{m x_1}.$$

The electron speed varies according to

$$v(t) = v_0 - \frac{e|U_m|}{m x_1}\, t,$$

where v_0 is the initial speed of the electron at the cathode surface. For the electron to reach the anode, its speed at the point $x = x_1$ must be greater than or equal to zero (since on the interval $x_1 \le x \le x_2$ the electron will be accelerated). The minimal energy of such an electron is most easily found from the conservation of energy:

$$\frac{m v_{0\,\min}^2}{2} = e|U_m| = 3.6 \cdot 10^{-19}\ \text{J}.$$

The flight time τ_1 through the interval $0 \le x \le x_1$ in this case is obtained from the condition

$$v_{0\,\min} - \frac{e|U_m|}{m x_1}\, \tau_1 = 0,$$

whence

$$\tau_1 = \frac{m x_1 v_{0\,\min}}{e|U_m|} = \frac{m x_1 \sqrt{2e|U_m|}}{e|U_m|\sqrt{m}} = \frac{\sqrt{2}\, x_1}{\sqrt{e|U_m|/m}}.$$

The motion of the electron on the interval $x_1 \le x \le x_2$ is uniformly accelerated, with acceleration

$$a_2 = \frac{e\,(U_a + |U_m|)}{m\,(x_2 - x_1)}.$$

Since the electron's initial speed at $x = x_1$ equals zero, the time of flight through this interval

is

$$\tau_2 = \sqrt{\frac{2\,(x_2 - x_1)}{a_2}} = \frac{\sqrt{2}\,(x_2 - x_1)}{\sqrt{e\,(U_a + |U_m|)/m}}.$$

The total time of flight is

$$\tau = \tau_1 + \tau_2 = \sqrt{\frac{2}{\gamma}}\left(\frac{x_1}{\sqrt{|U_m|}} + \frac{x_2 - x_1}{\sqrt{U_a + |U_m|}}\right) = 10^{-8}\ \text{s}.$$

PROBLEM 29

Two Metallic Rings

From Kvant, *1993, No. 6*

Veronika's art
p29.png

Problem. Two fixed, identical, thin metallic rings are aligned coaxially at some distance from each other (see Fig. 29.1). The rings carry charges equal in magnitude but opposite in sign. For a charged particle to pass along the straight line that goes through the centers of the rings and is perpendicular to their planes, the particle must have a minimal speed v_0 when it is far from the rings. Let the particle's speed far from the rings be v_1 ($v_1 > v_0$). What is the ratio of the particle's maximum speed to its minimum speed during the flyby?

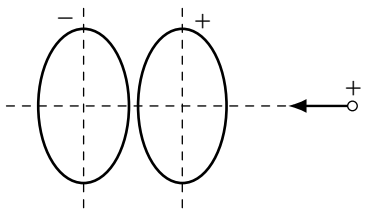


Fig. 29.1

Solution. A qualitative distribution of the electrostatic potential $\varphi(r)$ of the charged rings along the particle's line of motion is shown in Fig. 29.2. The origin is chosen midway between the rings; the coordinate of the positively charged ring is $r_1 = a$, and that of the negatively charged ring is $r_2 = -a$. In this problem the specific analytic form of $\varphi(r)$ is unimportant; what matters is that $\varphi(r)$ is an odd function, $\varphi(r) = -\varphi(-r)$, and that it possesses two extrema: a maximum and a minimum.

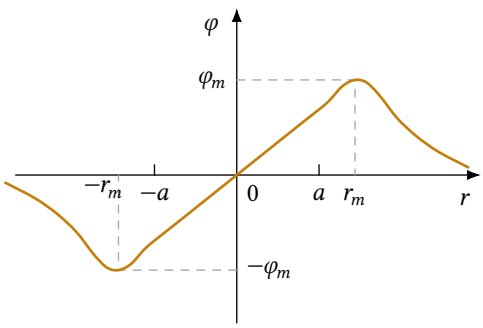


Fig. 29.2

A positively charged particle flying from right to left is decelerated on the intervals $r_m < r < \infty$ and $-\infty < r < -r_m$, and accelerated on the interval $-r_m < r < r_m$. The minimal speed v_0 required for the particle to pass through the rings is determined by the condition

$$\frac{mv_0^2}{2} = q\,\varphi_m,$$

where m is the particle's mass and q is its charge. In this case, at the point $r = r_m$ the particle's speed is zero; then it accelerates and at $r = 0$ its speed again equals v_0. Continuing to accelerate, at $r = -r_m$ its speed reaches a maximum equal to $\sqrt{2}\,v_0$. On the interval $-\infty < r < -r_m$ the speed begins to decrease, approaching v_0 far away.

From this reasoning it is clear that, if the initial speed is v_1, the particle attains its minimal speed at $r = r_m$ and its maximal speed at $r = -r_m$. Both speeds follow from energy conservation:

$$\frac{mv_1^2}{2} = \frac{mv_{\min}^2}{2} + q\varphi_m = \frac{mv_{\min}^2}{2} + \frac{mv_0^2}{2},$$

$$\frac{mv_1^2}{2} = \frac{mv_{\max}^2}{2} - q\varphi_m = \frac{mv_{\max}^2}{2} - \frac{mv_0^2}{2}.$$

From the first equation we obtain

$$v_{\min} = \sqrt{v_1^2 - v_0^2}\,,$$

and from the second,

$$v_{\max} = \sqrt{v_1^2 + v_0^2}\,.$$

Their ratio is

$$\frac{v_{\max}}{v_{\min}} = \sqrt{\frac{1 + (v_0/v_1)^2}{1 - (v_0/v_1)^2}}\,.$$

PROBLEM 30

Pseudo-experiment “Viscous Friction”

From the All-Russian Olympiad, regional stage, 2021



Problem. A falling paper cone experiences a viscous air-drag force F_{drag} that depends on the cone's speed:

$$F_{\text{drag}} = k\, v^n.$$

Study the fall of a paper cone for different total masses and determine the coefficient n. Estimating the uncertainty is not required.

Remark. In mathematics the logarithm function is often used; it is the inverse of exponentiation. From its properties, if $F_{\text{drag}} = k\, v^n$, then

$$\ln\bigl(F_{\text{drag}}\bigr) = \ln(k) + n\, \ln(v).$$

The value of a logarithm can be computed on a calculator.

Equipment. Two modular photodetectors, a power supply, electronic scales, a USB oscilloscope, a laptop, a plumb line, thread, a ruler, scissors, glue, a cone template, plasticine, a tissue, and millimeter graph paper for plotting graphs.

Description of the setup

The setup makes it possible to determine with high accuracy the fall time of a paper cone over a section where its motion is uniform. To measure this time, a system of two modular photodetectors (the upper one 4 and the lower one 5) is used; their signal is sent via a USB oscilloscope 6 to a laptop 7.

The upper modular photodetector consists of a laser 13 and a photosensor 12. When the cone crosses the laser beam, the voltage on the photosensor changes.

The lower modular photodetector is a square box whose two opposite inner walls are covered with plane mirrors. A laser 10 is fixed on a side wall of the box; its beam, after multiple reflections from the mirrors, hits photosensor 11 while remaining in one horizontal plane the whole time. When the cone falls inside the box, the laser beam is guaranteed to be interrupted and the voltage on the photosensor changes.

The distance from the beam of the upper laser to the plane formed by the beams of the lower laser equals

$$L = (23.0 \pm 0.5)\ \text{cm}.$$

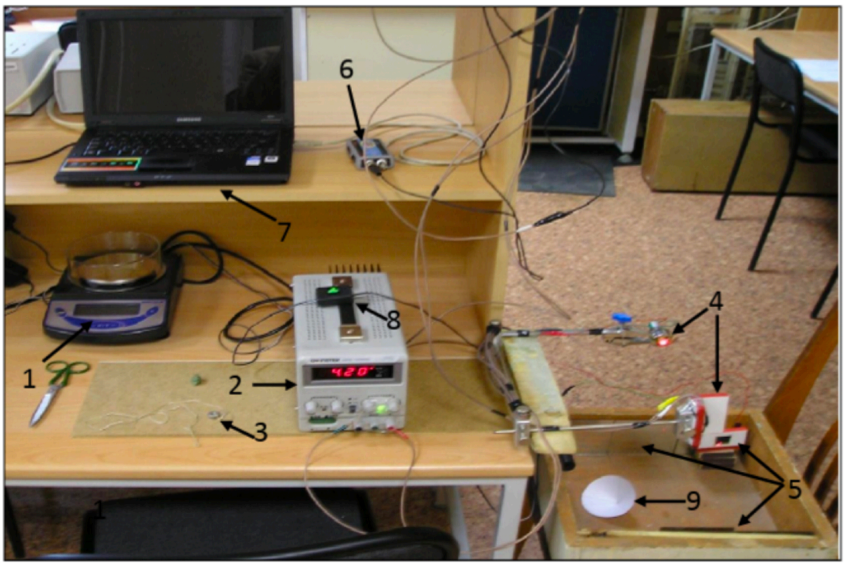


Fig. 30.1

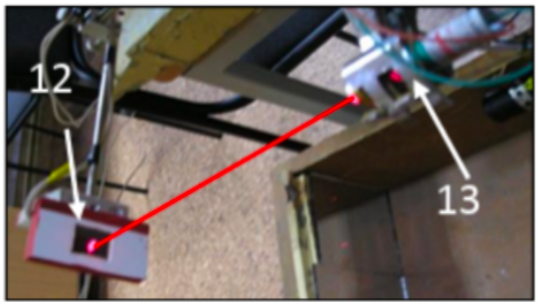


Fig. 30.2

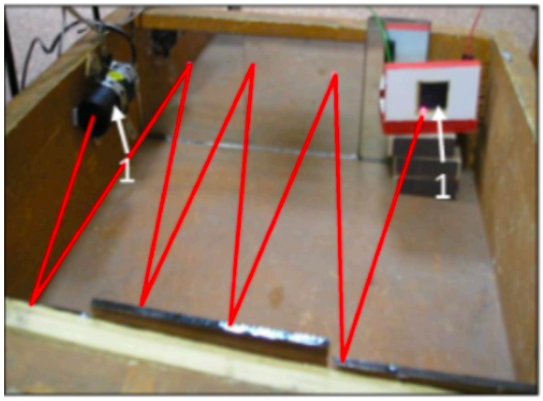


Fig. 30.3

Photosensors 11 and 12 are connected in series in an electric circuit and to the USB oscilloscope, which converts the total voltage on the photosensors into digital form and sends it to the laptop 7. Special software is installed on the laptop to display a graph of voltage versus time with a resolution of up to 0.5 ms. From the resulting graph one can determine the motion time of the cone between the two laser beams. The cone mass is determined using the electronic scales 1. To change the mass, a small piece of plasticine is placed inside the cone.

Measurement procedure

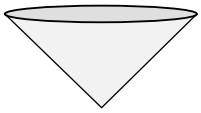


Fig. 30.4

1. Place some plasticine inside the cone and determine on the electronic scales the total mass of the cone with the plasticine.
2. Position the cone tip-down at a distance of approximately 60–70 cm above the upper laser and release it. A distance of 60–70 cm guarantees that when approaching the upper laser its motion will be practically uniform.
3. In the program window, from the dependence of total voltage on time, determine and enter into a table the seven parameters:

   T_1, T_2 — start and end times of the first dip; T_3, T_4 — start and end times of the second dip;

   U_0 — constant voltage level; U_1, U_2 — voltage levels corresponding to the first and second dips.

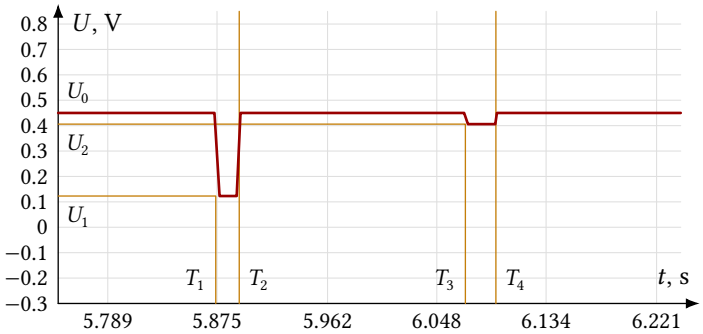


Fig. 30.5

4. Repeat the experiments with a different mass of plasticine inside the cone.

MEASURED DATA.

Mass* , mg	T_1, ms	T_2, ms	T_3, ms	T_4, ms	U_0, V	U_1, V	U_2, V
533	161	181	372	386	0.450	0.123	0.406
600	128	143	331	341	0.450	0.136	0.407
700	192	198	372	381	0.450	0.127	0.402
790	546	552	715	723	0.450	0.123	0.410
894	133	149	289	301	0.450	0.126	0.402
990	120	131	257	277	0.450	0.122	0.392
1125	570	576	694	709	0.450	0.122	0.403

* Mass of the cone with plasticine.

Solution. First determine which data are needed and how to process them. The statement says that after release the cone's motion on the section between the sensors may be considered uniform. In this case the weight of the cone is balanced by the drag force, so we can write

$$F_{\text{drag}} = mg,$$

where m is the mass of the cone with plasticine.

Since the motion is uniform, the speed can be calculated as

$$v = \frac{L}{t},$$

where L is the distance between the sensors and t is the flight time between the sensors. From the statement,

$$L = (23.0 \pm 0.5)\ \text{cm}.$$

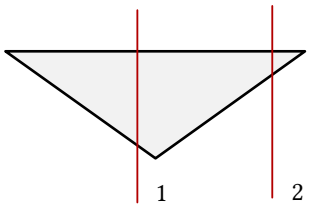


Fig. 30.6

It remains to interpret the times. The table contains four time marks, and we must understand which are needed. Consider how the cone crosses a laser beam. From the sketch (side view of the cone), two possible trajectories of the laser beam relative to the cone are shown. From this it is clear that one cannot tell at what height the cone was when it *began* to block the laser beam; however, the moment when the laser beam *ceases* to be blocked by the cone uniquely determines the cone position. Therefore, the required times are T_2 and T_4. Hence,

$$v = \frac{L}{T_4 - T_2}.$$

From the statement we know that $F_{\text{drag}} = k\,v^n$, therefore

$$\ln\big(F_{\text{drag}}\big) = \ln(k) + n\,\ln(v).$$

m, mg	T_2, ms	T_4, ms	F_{drag}, 10^{-3} N	v, m/s	$\ln F_{\text{drag}}$	$\ln v$
533	181	386	5.22	1.122	−5.25	0.115
600	143	341	5.88	1.162	−5.14	0.150
700	198	381	6.86	1.257	−4.98	0.229
790	552	723	7.74	1.345	−4.86	0.296
894	149	301	8.76	1.513	−4.74	0.414
990	131	277	9.70	1.575	−4.64	0.454
1125	576	709	11.03	1.729	−4.51	0.548

Fig. 30.7

To determine the parameter n we plot a graph in logarithmic coordinates ($\ln F_{\text{drag}}$ versus $\ln v$).

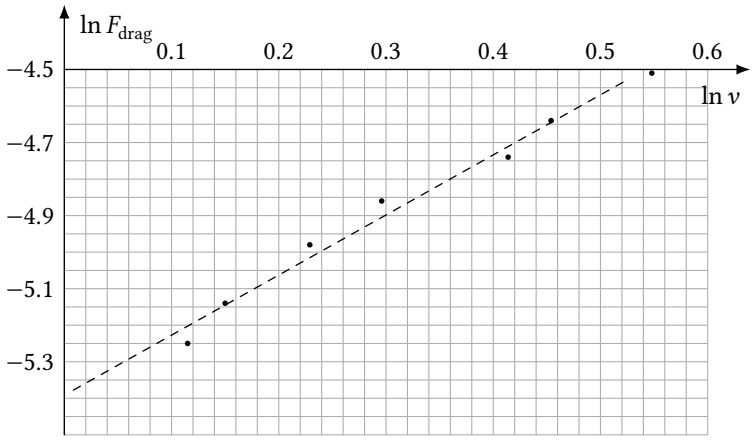


Fig. 30.8

From the graph we find the slope (and the intercept with the vertical axis). We obtain

$$n = 1.64.$$

IV

PART IV

Problems 31–40

31 Triangle and Film
32 Rotating Sleeve
33 Adiabatic Anisotropy
34 Silvered Cone
35 Current Stabilizer
36 Surely Not Snellius?
37 I Put on My Glasses
38 The Research Probe
39 Impulse Acceleration
40 Lord Kelvin's Archive

PROBLEM 31

Triangle and Film

From the All-Russian Olympiad, regional stage, 2022

Veronika's art
p31.png

Problem. Light rods OA and AB are connected to each other by hinges. The end O of rod OA is hinged to a smooth spoke, and at the end B of rod AB a small ring of mass m is attached by a hinge; the ring can slide along the spoke. The rod lengths differ by a factor of two: $|AB| = L$, $|OA| = 2L$. All hinges are weightless. The system is surrounded on the outside (up to the fixed outer boundary) by a two-sided film with surface tension coefficient σ. There is no film in the region between the spoke and the rods. Neglect gravity.

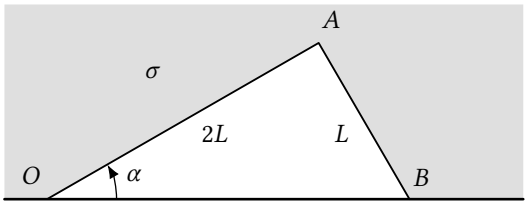


Fig. 31.1

1. Find the value of the angle α in the equilibrium position.
2. Find the period of small oscillations of the system near the equilibrium position.

A possible solution. Method 1 (energetics).

1) In equilibrium the energy of the system must be minimal. Here the system's energy is the energy of the stretched film, which is proportional to the area of the film; taking into account that the film is two-sided,

$$E_{\text{fl}} = 2\sigma S_{\text{fl}}.$$

It is clear that the minimum of this energy corresponds to the position of the rods in which the area of triangle OAB is maximal. (The film fills the whole region outside the triangle, so its energy is $E = 2\sigma(S - S_{OAB})$, where S is the total area inside the outer boundary.)

It is convenient to compute its area from two sides (OA and AB) and the angle φ between them:

$$S_{OAB} = \frac{1}{2} \cdot L \cdot 2L \cdot \sin(\varphi) = L^2 \sin(\varphi).$$

This area is maximal at $\varphi = \dfrac{\pi}{2}$, i.e. when the rods are perpendicular to each other, and

therefore in the right triangle OAB

$$\tan(\alpha) = \frac{1}{2}.$$

Thus, in equilibrium

$$\alpha = \arctan\left(\frac{1}{2}\right) = \arcsin\left(\frac{1}{\sqrt{5}}\right) \approx 26.6^\circ.$$

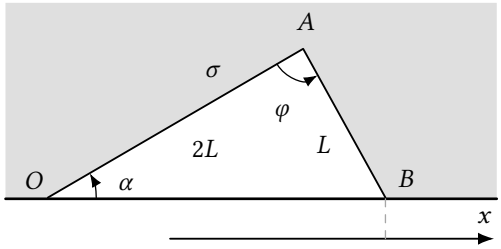


Fig. 31.2

2) Potential energy is defined up to an additive constant, so we can set the energy in equilibrium to zero. Then the energy of the system in a configuration where the ring is shifted along the spoke to a point with coordinate x, measured from the equilibrium position (see the figure), equals

$$E(x) = 2\sigma\big[L^2 - S_{OAB}(x)\big] = 2\sigma L^2\{1 - \sin[\varphi(x)]\}.$$

The length of side OB in this configuration is $|OB| = L\sqrt{5} + x$, and by the law of cosines

$$(L\sqrt{5} + x)^2 = L^2 + 4L^2 - 4L^2\cos[\varphi(x)] \;\Rightarrow\; \cos[\varphi(x)] \approx -\frac{\sqrt{5}}{2L}\,x.$$

Note that in this expression we neglected terms of order x^2/L^2 (small oscillations).

In the same approximation

$$\sin[\varphi(x)] \approx \sqrt{1 - \frac{5x^2}{4L^2}} \approx \left(1 - \frac{5x^2}{4L^2}\right)^{\frac{1}{2}} \approx 1 - \frac{5x^2}{8L^2}.$$

Hence,

$$E(x) \approx \frac{5\sigma x^2}{4} = \frac{kx^2}{2}.$$

Here we introduced $k = \dfrac{5\sigma}{2}$. We obtained exactly the same form as for a mass on a spring: the ring performs small harmonic oscillations with period

$$T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{2m}{5\sigma}}.$$

$$\boxed{\alpha = \arctan\left(\frac{1}{2}\right) = \arcsin\left(\frac{1}{\sqrt{5}}\right) \approx 26.6^\circ, \qquad T = 2\pi\sqrt{\frac{2m}{5\sigma}}.}$$

Another possible solution. Method 2 (statics and dynamics).

1) Let us consider the equilibrium of the rod AB, on which three forces act: the resultant of the surface-tension forces (perpendicular to the rod and applied at its midpoint), the force from the ring, and the reaction of hinge A. The resultant of the surface-tension forces which is perpendicular to the rod and applied at its midpoint. Since the rod is weightless, from the condition of equality of moments about the ring we find the projection of the hinge reaction force onto the direction perpendicular to the rod:

$$2\sigma L\frac{L}{2} - R_{AB\perp}L = 0 \Rightarrow R_{AB\perp} = \sigma L.$$

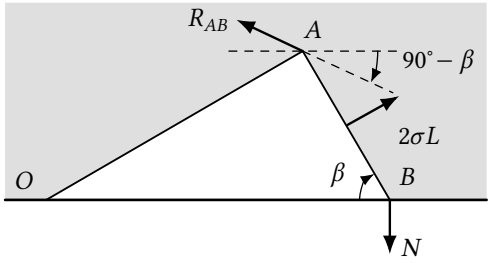


Fig. 31.3

Finally, from the condition of balance of the force projections on the rod, we find the component of the hinge reaction parallel to the rod:

$$R_{AB\parallel} = N\sin(\beta) = \sigma L\,\mathrm{tg}(\beta).$$

Thus, the magnitude of this force is

$$R_{AB} = \frac{\sigma L}{\cos(\beta)},$$

and it is directed at an angle $(2\beta - 90^\circ)$ to the rod, and at an angle $(90^\circ - \beta)$ to the horizontal.

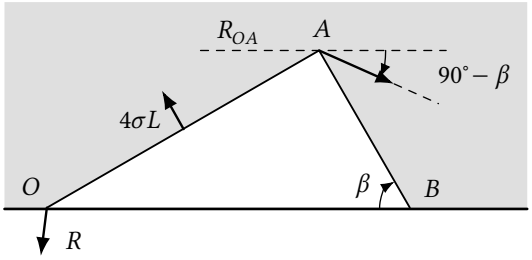


Fig. 31.4

The force acting on hinge A from the side of the rod AB, in accordance with Newton's third law, is equal in magnitude and opposite in direction to the force $\vec{R}_{AB}$ found above, as well as to the force acting on hinge A from the side of the rod OA (hinge A is in equilibrium under the action of these two forces). And, of course, the force $\vec{R}_{OA}$ acting from hinge A on the rod OA turns out to be equal in magnitude and opposite in direction to the force found

by us:

$$\vec{R}_{OA} = -\vec{R}_{AB}.$$

Now we can consider the equilibrium of the rod OA under the action of $\vec{R}_{OA}$, the resultant of the surface-tension forces, and the reaction force of hinge O. Writing the condition of equilibrium of moments about O, we find the component of $\vec{R}_{OA}$ perpendicular to the rod:

$$4\sigma L\frac{L}{2} - R_{OA\perp}L = 0 \Rightarrow R_{OA\perp} = 2\sigma L.$$

But from what was found earlier we obtain another expression for this quantity:

$$R_{OA\perp} = \frac{\sigma L}{\cos(\beta)}\cos(90^\circ - \beta) = \sigma L \cdot \mathrm{tg}(\beta).$$

By comparing these expressions we see that, in equilibrium, $\mathrm{tg}(\beta) = 2$. From the law of sines,

$$\sin(\alpha) = \frac{1}{2}\sin(\beta) = \frac{1}{\sqrt{5}},$$

and therefore $\alpha = \arcsin\left(\frac{1}{\sqrt{5}}\right) \approx 26.6^\circ$. Note that from the obtained angles ($\sin(\alpha) = \cos(\beta)$) it follows that the third angle in triangle OAB is a right angle!

2) Let us now consider a nonequilibrium position of the ring, in which it has coordinate x, measured from the equilibrium position (see the figure).

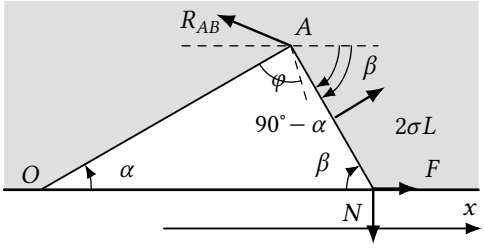


Fig. 31.5

Next we shall assume that the notation of all angles refers specifically to the displaced position of the ring. Since we need to describe small harmonic oscillations, all forces must be calculated to first order in x/L. For example, the length OB in this position is $|OB| = L\sqrt{5}+x$, and, in accordance with the law of cosines,

$$(L\sqrt{5}+x)^2 = L^2 + 4L^2 - 4L^2\cos(\varphi) \Rightarrow \cos(\varphi) \approx -\frac{\sqrt{5}}{2L}x.$$

Accordingly, to this accuracy,

$$\sin(\varphi) \approx \sqrt{1 - \frac{5x^2}{4L^2}} \approx 1.$$

By the law of sines,

$$\sin(\beta) = \frac{2L}{L\sqrt{5}+x}\sin(\varphi) \approx \frac{2}{\sqrt{5}} - \frac{2}{5L}x \Rightarrow \cos(\beta) \approx \frac{1}{\sqrt{5}} + \frac{4}{5L}x.$$

Since the rods and hinges are weightless, the sum of the forces and the sum of the moments of forces applied to each of them are zero. In the rest position, the ring simply transmitted the normal reaction force of the spoke to the rod AB. Since the ring is massive and now moves with acceleration, the force acting from it on the rod AB has a component F along the spoke. Let us repeat the reasoning for the rod AB for the new situation. The condition of equilibrium of moments about hinge A gives equation (1):

$$2\sigma L\frac{L}{2} - NL\cos(\beta) + FL\sin(\beta) = 0 \Rightarrow NL\cos(\beta) - FL\sin(\beta) = \sigma L. \quad (1)$$

The condition of equilibrium of moments about the ring is the same:

$$2\sigma L\frac{L}{2} - R_{AB\perp}L = 0 \Rightarrow R_{AB\perp} = \sigma L.$$

The condition of force balance in the projection onto the rod AB has now become

$$R_{AB\parallel} = N\sin(\beta) + F\cos(\beta).$$

The condition of equilibrium of moments of forces acting on the rod OA about O has not changed either:

$$4\sigma L\frac{L}{2} - R_{OA\perp}L = 0 \Rightarrow R_{OA\perp} = 2\sigma L.$$

Taking into account $\vec{R}_{OA} = -\vec{R}_{AB}$, we can compute this component via the forces acting on AB (it is important to note that the direction perpendicular to OA makes an angle $(90^\circ - \alpha)$ with the horizontal, while the rod AB makes an angle β):

$$R_{OA\perp} = 2\sigma L = R_{AB\parallel}\cos(90^\circ - \alpha - \beta) + R_{AB\perp}\cos(180^\circ - \alpha - \beta).$$

With $(180^\circ - \alpha - \beta = \varphi)$ and the expressions for the forces, we obtain equation (2):

$$N\sin(\beta) + F\cos(\beta) = \sigma L\frac{2 - \cos(\varphi)}{\sin(\varphi)} \approx \sigma L\left(2 + \frac{\sqrt{5}}{2L}x\right). \quad (2)$$

Eliminating the force N from (1) and (2), and using the approximate formulas for the angles obtained above, we find that

$$F \approx \sigma L\left[\left(2 + \frac{\sqrt{5}}{2L}x\right)\cos(\beta) - \sin(\beta)\right] \approx \frac{5}{2}\sigma \cdot x.$$

The corresponding component of the force acting from the rod AB on the ring has the opposite sign. Thus, the equation of motion of the ring for $x \ll L$ has the form

$$m\ddot{x} \approx -\frac{5}{2}\sigma \cdot x.$$

This is the equation of harmonic oscillations with angular frequency

$$\omega = \sqrt{\frac{5\sigma}{2m}}.$$

The oscillation period is

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{2m}{5\sigma}}.$$

$$\boxed{\alpha = \arctan\left(\frac{1}{2}\right) = \arcsin\left(\frac{1}{\sqrt{5}}\right) \approx 26.6^\circ, T = 2\pi\sqrt{\frac{2m}{5\sigma}}.}$$

PROBLEM 32

Rotating Sleeve

From the All-Russian Olympiad, regional stage, 2023

Veronika's art
p32.png

Problem. A thin-walled cylindrical sleeve of mass m, rotating with angular speed ω_0 about its own axis, flies with speed v_0 into a circular hole in a steel plate (left-hand figure). The axes of the sleeve and the hole coincide, the outer radius of the sleeve R equals the radius of the hole, and the sleeve length l equals the thickness of the plate.

The graph of the force required to push a *non-rotating* sleeve through the hole as a function of the sleeve displacement is shown in the right-hand figure. The maximum value of the force equals F_0. This force is that needed to overcome dry friction; the normal reaction forces exerted by the hole walls on the sleeve surface elements do not depend on the (translational and angular) speeds of the sleeve. All surfaces of the sleeve and the hole are homogeneous and identical along the entire length. The coordinate $x = 0$ corresponds to the position at which the sleeve has just started to enter the plate.

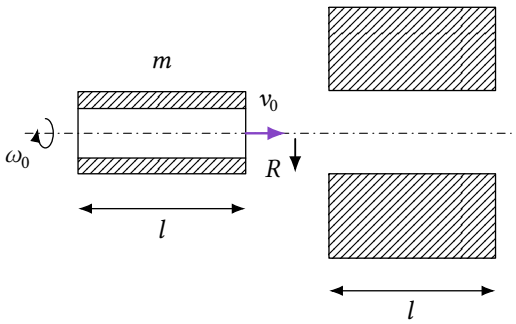


Fig. 32.1

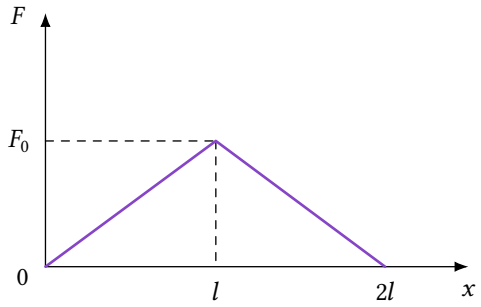


Fig. 32.2

1. For what minimum value $v_0 = v_{\min}$ will the sleeve pass through the hole (the initial angular speed ω_0 is fixed)?
2. What will be, in this case (for $v_0 = v_{\min}$), the angular speed ω_1 of the sleeve when it is entirely inside the plate?
3. After what time τ from the moment it enters the hole with initial speed $v_0 \geqslant v_{\min}$ will the sleeve exit completely from the plate?

Solution. The velocity of points on the outer surface of the sleeve makes with the sleeve axis an angle

$$\alpha = \operatorname{arctg} \frac{\omega R}{v}.$$

The friction force acting on surface elements of the sleeve is directed opposite to these velocities and therefore does not change the angle between the velocities of surface points of the sleeve and its axis, but it does affect the magnitudes of these velocities. Consequently, during the braking of the sleeve the ratio of the translational speed to the linear speed of rotation of the surface points remains unchanged:

$$\frac{\omega R}{v} = \frac{\omega_0 R}{v_0}.$$

Since this circumstance is very important for the solution, we also give another justification. Since the normal reaction of the hole walls is constant (as given), the kinetic friction force on each small element of the sleeve surface has constant magnitude and is directed opposite to the motion. The action of the friction can be split into the braking of translation (resulting force $F_x(x) = F(x)\cos\alpha$) and the braking of rotation (braking torque equals the moment of the force $F_\perp(x) = F(x)\sin\alpha$). Therefore, the decelerations with which the speed of its translational motion $v_x = v\cos\alpha$ and the linear speed of the rotational motion of points on its surface $v_\perp = \omega R = v\sin\alpha$ decrease are proportional to these speeds. Hence, the ratio of these speeds remains unchanged:

$$\frac{(v_\perp)'_t}{(v_x)'_t} = \frac{v_\perp}{v_x}, \qquad \left(\frac{v_\perp}{v_x}\right)'_t = \frac{v_x(v_\perp)'_t - v_\perp(v_x)'_t}{v_x^2} = 0, \qquad \frac{v_\perp}{v_x} = \text{const}.$$

Therefore $\operatorname{tg}\alpha = \dfrac{v_\perp}{v_x} = \text{const}$.

In the direction of translation the acceleration of the sleeve is determined by the projection of the friction force:

$$F_x(x) = F(x)\cos\alpha = F(x)\,\frac{v_0}{\sqrt{(\omega_0 R)^2 + v_0^2}}.$$

Therefore, the motion of the *rotating* sleeve along the axis is the same as the motion of a non-rotating sleeve under a "reduced" force $F_x(x)$. The minimum speed $v_{\min}$ accounting for this can be found from energy conservation:

$$\frac{mv_{\min}^2}{2} = -A_{\text{mp}} = \frac{F_x(l)\,2l}{2} = \frac{F_0 l\, v_{\min}}{\sqrt{(\omega_0 R)^2 + v_{\min}^2}}.$$

From this equation we obtain

$$v_{\min} = \sqrt{\sqrt{\frac{(\omega_0 R)^4}{4} + 4\left(\frac{F_0 l}{m}\right)^2} - \frac{(\omega_0 R)^2}{2}}\,.$$

To answer the second question it suffices to note that the angular speed and the translational speed are related at all times by

$$\frac{\omega}{v} = \frac{\omega_0}{v_0}.$$

At the instant of complete immersion the translational speed v_1 is smaller by a factor $\sqrt{2}$ than the initial speed $v_0 = v_{\min}$, since the work of the friction forces by that moment equals exactly one half of the work up to the exit moment. Therefore

$$\omega_1 = \frac{v_1}{v_0}\,\omega_0 = \frac{\omega_0}{\sqrt{2}}.$$

While the sleeve is inside the hole (before it is fully inside the plate), the force F_x increases linearly with the penetration depth. The sleeve acceleration is

$$a = -\frac{F_0\cos\alpha}{ml}\,x.$$

This is the equation of harmonic oscillations with angular frequency

$$\Omega = \sqrt{\frac{F_0\cos\alpha}{ml}} = \sqrt{\frac{F_0 v_0}{ml\sqrt{(\omega_0 R)^2 + v_0^2}}}.$$

The coordinate of the leading edge of the sleeve depends on time as

$$x = x_0 \sin\Omega t,$$

where

$$x_0 = \frac{v_0}{\Omega} = v_0\sqrt{\frac{ml}{F_0\cos\alpha}}.$$

Hence, up to the instant of complete immersion of the sleeve in the hole,

$$x = v_0\sqrt{\frac{ml}{F_0\cos\alpha}}\,\sin\Omega t,$$

and the immersion time τ is given by

$$l = v_0\sqrt{\frac{ml}{F_0\cos\alpha}}\,\sin\Omega t, \qquad \tau = \sqrt{\frac{ml}{F_0\cos\alpha}}\,\arcsin\left(\sqrt{\frac{F_0 l\cos\alpha}{mv_0^2}}\right).$$

Equivalently,

$$\tau = \sqrt{\frac{ml\sqrt{(\omega_0 R)^2 + v_0^2}}{F_0 v_0}}\,\arcsin\left(\sqrt{\frac{F_0 l}{mv_0\sqrt{(\omega_0 R)^2 + v_0^2}}}\right).$$

PROBLEM 33

Adiabatic Anisotropy

From the All-Russian Olympiad, final stage, 2023

Veronika's art
p33.png

Problem. A horizontal cylindrical vessel is sealed with a piston that can be moved by an attached handle. The vessel contains *saturated* water vapor at temperature $T_0 = 333$ K. There is no liquid water in the vessel. Water vapor may be treated as an ideal polyatomic gas. The specific latent heat of vaporization of water at temperature T_0 is $L = 2.36$ MJ/kg and within the scope of the problem it may be considered temperature-independent. The universal gas constant is $R = 8.31$ J/(mol·K), and the molar mass of water is $\mu = 18.0$ g/mol.

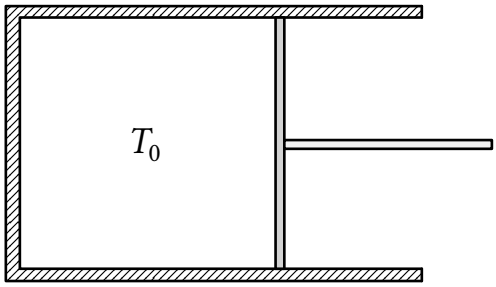


Fig. 33.1

Assume that small relative changes of the pressure of saturated vapor and of its absolute temperature near the values $p_0(T_0)$ and T_0 are related by

$$\varepsilon_p = \frac{\Delta p}{p_0} = \alpha\, \varepsilon_T = \alpha \cdot \frac{\Delta T}{T_0}, \qquad \alpha = 15.3.$$

1. The temperature in the vessel is slowly varied. The vessel volume is changed in such a way that all substance in the vessel remains in the gaseous state at all times, while the vapor is saturated all the time. What is the molar heat capacity of the water vapor in this process?

   Consider now an adiabatically insulated vessel.

2. Find the temperature change ΔT_1 in the vessel for a slow relative decrease of its volume by $\beta = 5\%$.

3. Find the temperature change ΔT_2 in the vessel for a slow relative increase of its volume by $\beta = 5\%$.

Solution. 1.

Write the first law of thermodynamics for the substance under the piston:

$$\delta Q = dU + \delta A = dU + p\,dV = dU + d(pV) - V\,dp.$$

Next, use the expression for the internal energy of an ideal gas and the Clapeyron–Mendeleev equation:

$$U = \nu C_V T, \qquad pV = \nu RT.$$

Since the substance in the vessel remains entirely in the gaseous phase,

$$d(pV) = \nu R\,dT.$$

Hence we obtain

$$\delta Q = \nu C_V\,dT + \nu R\,dT - V\,dp = \nu C_p\,dT - V\,dp,$$

where $C_p = C_V + R = 4R$ is the molar heat capacity of an ideal polyatomic gas at constant pressure.

For the heat capacity of an ideal gas expressed via the parameters (p, T) we have:

$$C = \frac{\delta Q}{dT} = \nu C_p - V \cdot \frac{dp}{dT} = \nu C_p - \nu \frac{RT}{p} \cdot \frac{dp}{dT} = \nu \left(C_p - R \cdot \frac{\varepsilon_p}{\varepsilon_T} \right).$$

Therefore the molar heat capacity of the saturated vapor is

$$C_{\text{sat}} = C_p - \alpha R = R(4 - \alpha) = -93.9\ \text{J/(mol} \cdot \text{K)}.$$

2.

When the vessel volume changes, two principal physical scenarios are possible:

1. If the relative pressure change of the vapor under the piston is less than the relative pressure change of the saturated vapor for the same relative temperature change, i.e. if $\varepsilon_p < \varepsilon_p(\text{sat}) = \alpha\varepsilon_T$, then the substance in the vessel remains entirely in the gaseous state and the vapor ceases to be saturated.
2. If the stated relation between the relative pressure changes for the same relative temperature change is not satisfied, the vapor in the vessel remains saturated and partially condenses.

First, find the relative pressure change of the vapor under the piston assuming it remains entirely gaseous. From the Clapeyron–Mendeleev equation,

$$pV = \nu RT \;\Rightarrow\; p\,dV + V\,dp = \nu R\,dT = \frac{pV}{T}\,dT \;\Rightarrow\; \varepsilon_p + \varepsilon_V = \varepsilon_T.$$

Write the first law for an adiabatic process:

$$dU = -\delta A \;\Rightarrow\; \nu C_V\,dT = -p\,dV = -\nu RT\,\frac{dV}{V} \;\Rightarrow\; \varepsilon_T = -\frac{R}{C_V}\,\varepsilon_V.$$

Note that $\varepsilon_T > 0$ because $\varepsilon_V < 0$.

Next we obtain

$$\varepsilon_p = \varepsilon_T - \varepsilon_V = \varepsilon_T \left(1 + \frac{C_V}{R} \right) = C_p\,\varepsilon_T = 4\varepsilon_T < \alpha\varepsilon_T.$$

Thus the entire substance under the piston remains gaseous. The expression obtained for ε_T is therefore applicable, and the temperature increases by

$$\Delta T = T_0 \cdot \frac{\beta}{C_V/R} = \frac{\beta T_0}{3} \approx 5.6 \text{ K}.$$

3.

From the solution to item 2 it follows that upon an increase of volume without condensation the temperature in the vessel would have to decrease. Then the vapor in the vessel must remain saturated and partially condense; otherwise the vapor pressure in the vessel would exceed the saturated vapor pressure at the same temperature, which is impossible.

Differentiate the Clapeyron–Mendeleev equation taking into account the change in the gaseous amount of substance in the vessel:

$$pV = \nu RT \;\Rightarrow\; \varepsilon_p + \varepsilon_V = \varepsilon_\nu + \varepsilon_T \;\Rightarrow\; \varepsilon_T(\alpha - 1) + \varepsilon_V = \varepsilon_\nu,$$

since in the present process $\varepsilon_p = \alpha\varepsilon_T$.

From the first law of thermodynamics:

$$\delta Q_{\text{sat}} + L\,dm = 0,$$

where Q_{sat} is the heat received by the non-condensing saturated vapor, and $dm = \mu\,d\nu$ is the change in the mass of water vapor.

Because the state of non-condensing water vapor in the coordinates (p, T) lies on the phase-equilibrium curve, its heat capacity equals the heat capacity C_{sat} found in part 1; therefore for the heat increment we have

$$\delta Q_{\text{sat}} = C_{\text{sat}}\,dT.$$

Remark. Note that the formula

$$\delta Q_{\text{sat}} = C_V\,dT + p\,dV,$$

where dV is the vessel-volume change, is not correct here, because due to the phase transition the quantity dV is not equal to the change of the volume occupied by the non-condensing water vapor.

Returning to the first law for the system, we obtain:

$$\nu C_{\text{sat}}\,dT + \lambda\mu\,d\nu = 0 \;\Rightarrow\; \varepsilon_\nu = -\frac{C_{\text{sat}}\,T\,\varepsilon_T}{\mu\lambda}.$$

Hence,

$$\varepsilon_T(\alpha - 1) + \varepsilon_V = \varepsilon_\nu = -\frac{C_{\text{sat}}\,T\,\varepsilon_T}{\mu\lambda}.$$

Finally we find:

$$\Delta T = -T_0 \cdot \frac{\beta}{\alpha - 1 + \dfrac{(C_p - \alpha R)T_0}{\mu\lambda}} = -T_0 \cdot \frac{\beta}{\alpha - 1 + \dfrac{(4 - \alpha)RT_0}{\mu\lambda}} \approx -1.2 \text{ K}.$$

PROBLEM 34

Silvered Cone

From the All-Russian Olympiad, final stage, 2023

Veronika's art
p34 . png

Problem. A large opaque plate of thickness h contains a hole. The hole's lateral surface is a truncated cone with base diameters d and $D = 4d$. The axis of the conical hole is perpendicular to the plate; the lateral (conical) surface is silvered and ideally reflects incident light. A flat screen is placed parallel to the plate at distance L from it (see figure).

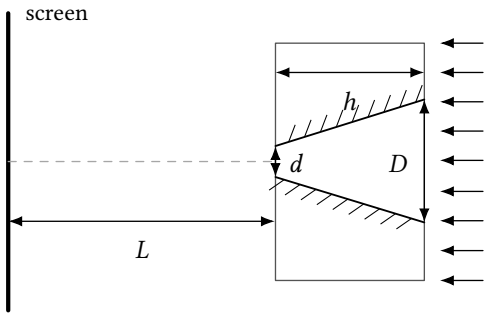


Fig. 34.1

The system is illuminated by a parallel beam of light directed along the cone axis. The geometric parameters satisfy $D \ll h \ll L$.

1. Draw the pattern observed on the screen. Indicate all characteristic geometric dimensions on your sketch. Assume light reaches the screen only through the hole in the plate.
2. An ideal thin converging lens is placed between the plate and the screen. The lens focal length is $F = L/2$; its optical axis coincides with the cone axis. The screen lies in the focal plane of the lens. Assume all rays emerging from the hole in the plate fall onto the lens. Draw the pattern observed on the screen in this case and indicate all characteristic geometric dimensions.

Note: Do not forget to include the constructions and calculations required to determine the characteristic geometric dimensions, as well as a justification of the appearance of the pattern observed on the screen.

Solution. 1. Consider a bundle of parallel rays incident on the right (in the figure) base of

the cone of diameter D. A portion of the rays, propagating within the circle of diameter d centered on the cone axis, passes through unobstructed and forms on the screen a central bright *circular spot* of the same diameter:

$$d_0 = d.$$

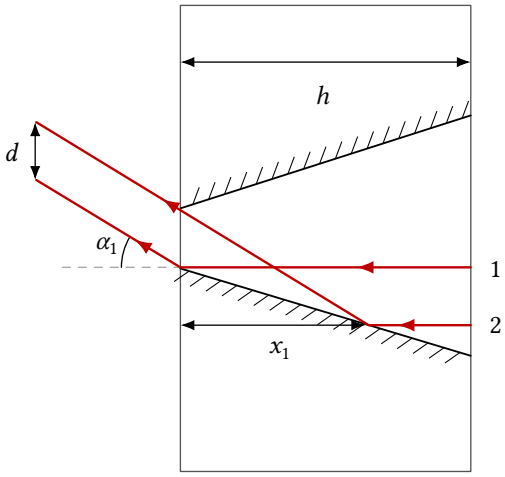


Fig. 34.2

All other rays that pass through the entrance cross–section of the cone undergo at least one reflection from the silvered lateral surface. Owing to axial symmetry, these rays form *rings* on the screen, centered at the intersection of the screen with the cone axis. Let us determine the number and geometry of the rings (their radii and widths, as well as the possibility of their overlapping with one another).

In what follows we examine rays that hit one of the generators of the conical surface. Let α_0 be the angle between the initial ray direction and the conical surface of the hole. By geometry,

$$\alpha_0 = \frac{D-d}{2h} = \frac{3d}{2h}, \quad \text{from the given condition:} \quad \alpha_0 \ll 1.$$

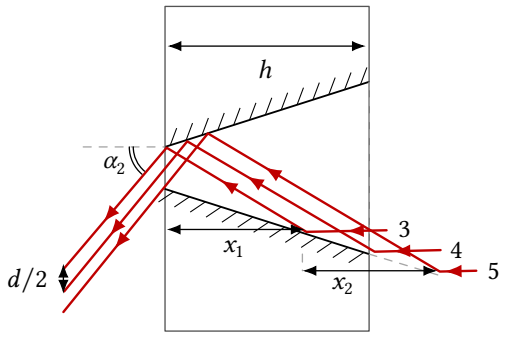


Fig. 34.3

Take ray 1 that strikes the very edge of the exit cross–section of the cone and has not yet

been reflected. After one reflection it leaves the hole at angle $2\alpha_0$ to the initial direction.

Next consider ray 2, incident on the conical surface as far as possible from the exit and undergoing exactly one reflection before leaving the plate. Let x_1 be the distance from the point where this ray hits the surface to the left face of the plate. Since $\alpha_0 \ll 1$, to good accuracy one may take the angle between the conical surface of the hole and the plate surface to be $\pi/2$. Then

$$x_1 \approx \frac{d}{\tan\alpha_0} = \frac{2h}{3}.$$

The rays of this type (with symmetry taken into account) produce a ring on the screen whose thickness is

$$d_1 = d.$$

The radius of the *midline* of this ring is found from the path of the ray that experienced one reflection and emerged from the center of the exit cross–section. Since this ray exits at angle $\alpha_1 = 2\alpha_0$ to the cone axis, the ring radius is

$$R_1 = L\tan\alpha_1 = \frac{3dL}{h}.$$

Note that $R_1 \gg d$ (the radius of the first ring is much larger than the diameter of the central spot), so the first ring does not overlap the central spot.

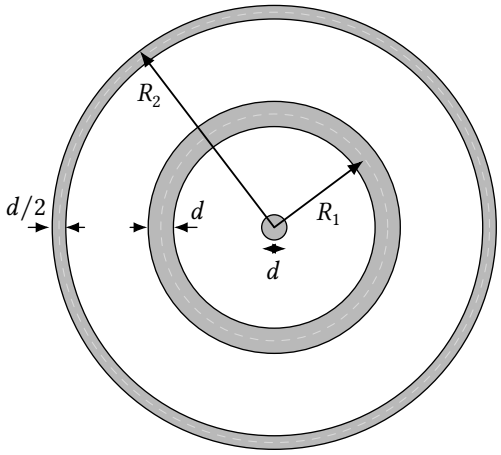


Fig. 34.4

Let us consider rays that are reflected twice from the conical surface of the aperture. They will leave the plate at an angle

$$\alpha_2 = 4\alpha_0$$

and will form one more ring on the screen. Let us find the distance x_2 between the points of the first reflection of the limiting rays in this group. Since $\alpha_0 \ll 1$, then with good accuracy

$$x_2 = x_1 = \frac{2h}{3}.$$

As we can see, the thickness of the plate is not sufficient for the rays that have undergone

two reflections to emerge in the given direction from the entire area of the exit aperture. From the paths of the limiting rays 3 and 4 it is seen that the thickness of the second ring is

$$d_2 = \frac{d}{2},$$

and the radius of the outer part of the second ring is

$$R_2 = L \operatorname{tg} \alpha_2 = \frac{6dL}{h}.$$

2.

All bundles of parallel rays exiting the hole are brought to focus in the focal plane. Their point of convergence can be located as follows: since $h \ll L$ and $d \ll L$, all rays may be considered as emerging practically from a single point coincident with the lens focus; hence, after refraction they propagate parallel to the optical axis. From the construction it is clear that in this case the *ring radii are halved*:

$$R_1' = \frac{R_1}{2} = \frac{3dL}{2h}, \qquad R_2' = \frac{R_2}{2} = \frac{3dL}{h}.$$

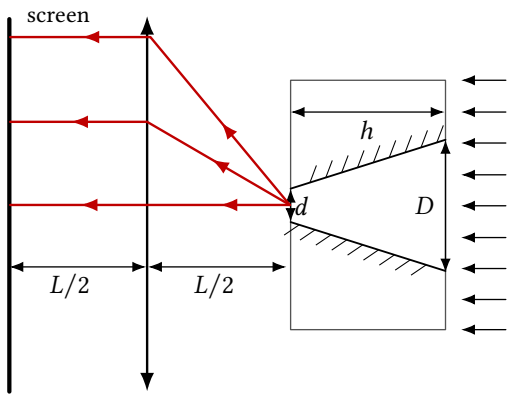


Fig. 34.5

Next, determine how the ring thicknesses and the diameter of the central spot change. The central spot is produced by the parallel bundle of rays propagating along the optical axis; an ideal lens focuses such rays to a point, therefore in the paraxial (geometric–optics) approximation the size of the central spot is much smaller than d.

The rings are also formed by *parallel* bundles which an ideal lens focuses in the focal plane; therefore the ring thicknesses also become much smaller than in part 1 (i.e. they become much less than d).

PROBLEM 35

Current Stabilizer

From the All-Russian Olympiad, regional stage, 2024

Veronika's art
p35 . png

Problem. While preparing for a physics olympiad, a tenth-grader Denis assembled the electric circuit (see Fig. 35.1) consisting of four identical ammeters, a diode, and an adjustable current source. The ammeter polarities and the direction of the current delivered by the source are shown on the diagram; the current–voltage characteristic of the diode is shown in Fig. 35.2. By varying the source current I_0, Denis found that the dependence of the current through one of the ammeters on I_0 has the shape shown in Fig. 35.3, where I_1 is a known parameter.

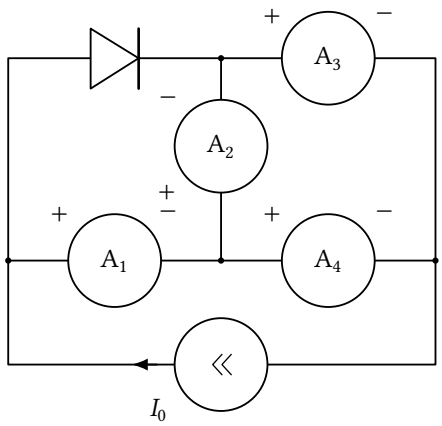


Fig. 35.1

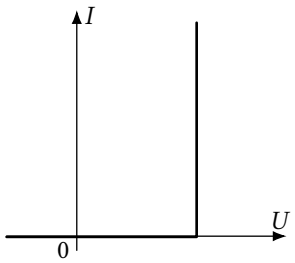


Fig. 35.2

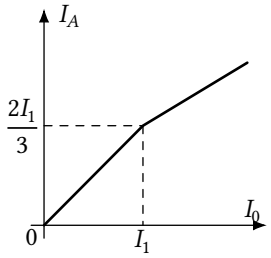


Fig. 35.3

1. For which of the ammeters (A_1, A_3, or A_4) is the dependence shown in Fig. 35.3 valid? Justify your answer.
2. Determine the value of I_0 at which no current flows through ammeter A_2.
3. Plot a qualitative graph of the reading of ammeter A_2 versus the source current, indicating on it the coordinates of the characteristic points.

Remark. A current source is a device through which a prescribed current flows, independent of the external circuit parameters.

Solution. 1.

Consider small values of the source current I_0 for which the diode is cut off. The equivalent circuit then has the form shown in the solution figure: the current through A_1 equals I_0. Since A_4 is connected in parallel to the pair of ammeters A_2 and A_3, its resistance is half the total resistance of the second and third instruments; therefore the current through A_4 is $2I_0/3$. The currents through A_2 and A_3 are then each equal to $I_0/3$.

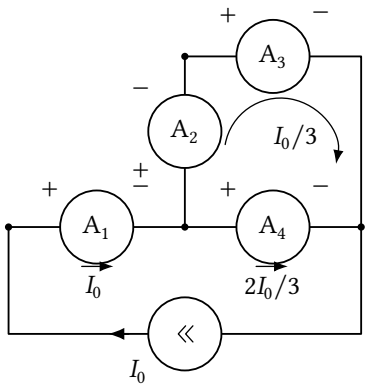


Fig. 35.4

Thus, the segment of the graph starting from the origin is described by the equation

$$I = \frac{2I_0}{3},$$

which coincides with the result obtained above for the fourth ammeter. Hence, the graph in Fig. 35.3 corresponds to ammeter A_4.

Let r be the internal resistance of an ammeter. We first determine the diode turn-on voltage U_D. Since the character of the dependence of the current through the fourth ammeter on I_0 changes at $I_0 = I_1$, this value corresponds to the instant when the voltage on the diode has already reached U_D, but the diode current is still zero (transition from the horizontal to the vertical section of the diode I–V curve). Write the equality of the voltage on the diode to the sum of the voltages on the first and second ammeters at $I_0 = I_1$:

$$U_D = I_1 r + \frac{I_1}{3} r = \frac{4I_1 r}{3}.$$

(Here the currents through A_1 and A_2 at $I_0 = I_1$ are I_1 and $I_1/3$, respectively.)

2.

The situation when no current flows through A_2 can occur only with the diode conducting (open). Moreover, if $I_{A2} = 0$, the currents through A_3 and A_4 are equal:

$$I_{A3} = I_{A4} = \frac{I_0}{2}.$$

Consequently, the voltages on the first ammeter and on the diode must be equal. Since in this case $I_{A1} = I_{A4}$, we have

$$U_D = I_{A1} r \;\Rightarrow\; \frac{4I_1 r}{3} = \frac{I_0}{2} r \;\Rightarrow\; \boxed{I_0 = \frac{8I_1}{3}}.$$

3.

While the diode is closed, i.e. for $I_0 < I_1$, the current through ammeter A_2 equals

$$I_{A2} = \frac{I_0}{3}.$$

If $I_0 > I_1$, the voltage across the diode is constant and equal to $U_D = 4I_1 r/3$. To plot the part of the desired graph that corresponds to the open diode, we can reason in two equivalent ways.

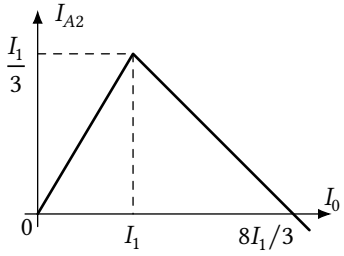


Fig. 35.5

Method 1. The diode I–V curve consists of two straight segments—horizontal and vertical. Therefore, if the operating point lies on only one segment (here, the vertical one), the diode can be treated as a linear element. The remaining elements of the circuit are linear

as well. Therefore, the dependence of the current in A_2 on I_0 for $I_0 > I_1$ is linear. Taking into account that this dependence must pass through the points $(I_1;\, I_1/3)$ and $\big(8I_1/3;\, 0\big)$, we obtain the straight line shown in the figure.

Method 2. Distribute the currents in the circuit for $I_0 > I_1$, taking into account that the voltage across the diode equals U_D. Let the current through A_2 be I_{A2}. Then from the equality of the sum of voltages on A_1 and A_2 to the diode voltage we have for the current through A_1:

$$I_{A1} = \frac{U_D - I_{A2}r}{r} = \frac{U_D}{r} - I_{A2}.$$

Accordingly, the currents through A_3 and A_4 are

$$I_{A4} = I_{A1} - I_{A2} = \frac{U_D}{r} - 2I_{A2}, \qquad I_{A3} = I_0 - I_{A4} = I_0 - \frac{U_D}{r} + 2I_{A2}.$$

Since the sum of voltages on A_2 and A_3 must equal the voltage on A_4,

$$I_{A2}r + I_{A3}r = I_{A4}r \;\Rightarrow\; \frac{U_D}{r} - 2I_{A2} = I_0 - \frac{U_D}{r} + 3I_{A2}.$$

Hence, using $U_D = \dfrac{4I_1 r}{3}$, we find

$$I_{A2} = \frac{2U_D}{5r} - \frac{I_0}{5} = \frac{8I_1}{15} - \frac{I_0}{5}.$$

Next, we plot the graphs of the dependences $I_{A2} = I_0/3$ for $I_0 < I_1$ and $I_{A2} = 8I_1/15 - I_0/5$ for $I_0 > I_1$, and we obtain the same figure as in Method 1.

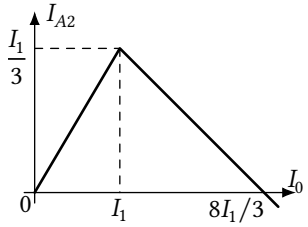


Fig. 35.6

PROBLEM 36

Surely Not Snellius?

From the All-Russian Olympiad, regional stage, 2025



Problem. A luminous segment AB is given, together with the point B' — the image of the point B formed by some thin lens, and a straight line α that passes through the point B' (see the figure). It is known that the point B' lies in the plane of the double focus of the lens, while the point A' (the image of the point A) lies on the line α. For all possible situations, reconstruct with a compass and an ungraduated ruler:

1. the position of the optical center of the lens;
2. the position of the lens and the position of its principal optical axis;
3. the positions of the foci of the lens;
4. the position of the point A'.

On a separate sheet two drawings are provided. Perform *all* constructions on that sheet and describe them. The standard geometric procedures such as drawing parallels and perpendiculars through a given point, bisecting a segment, copying equal segments, etc. need not be described in detail.

Assume that this lens refracts any rays (even non-paraxial ones) by the same rules as paraxial rays. A paraxial ray is a ray that propagates at a small angle to the principal optical axis of the lens and at a small distance from it.

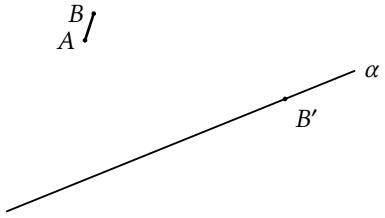


Fig. 36.1

Solution. The type of the lens is not specified in the statement, so we must determine what lens it can be. The image of a real object formed by a diverging lens always lies between the focal plane and the plane of the lens. Since B' lies in the double focal plane, the lens cannot be diverging. Therefore, the lens is *converging*. The image B' may be real or virtual.

First consider the case when the image B' is *real*. If the image is real, then the points B and B' lie in the double focal planes and the optical center lies at the middle of the segment BB'.

Now consider the case when the image B' is *virtual*. In this case B lies between the focal plane and the lens plane. Write the thin-lens formula for this case:

$$\frac{1}{a} + \frac{1}{-2F} = \frac{1}{F},$$

whence we find that $a = 2F/3$, and the distance between the object and its image is

$$\Delta = 2F - a = \frac{4F}{3},$$

which is twice the distance between the object and the lens a.

To construct the optical center of the lens: in the first (real-image) case bisect the segment BB'; the midpoint of this segment is the optical center O. In the second (virtual-image) case bisect the segment BB', extend the straight line through this segment beyond the point B, and mark off *another* half of the segment BB'; the obtained point is the optical center O.

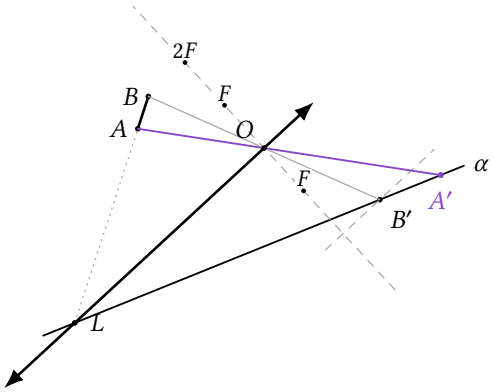


Fig. 36.2

To recover the position of the lens, find a second point of the lens. Note that the ray AB after refraction in the lens goes along the line α, since the image point A' also lies on this line. Extend the straight line passing through the segment AB until it intersects the line α; the found point of intersection L belongs to the plane of the lens. Draw the lens as a segment containing the points L and O. To reconstruct the principal optical axis (denoted below by POA), draw the perpendicular to the lens through the point O.

To find a focus one can proceed in two ways.

(a) Draw from the point B a ray parallel to POA. After refraction in the lens this ray passes through the point B' and intersects POA at the focus.

(b) Recall that the point B' lies in the plane of the double focus. Draw this plane, determine its intersection with POA (this is the double focus), bisect the obtained segment, and we get the (single) focal point F. The drawings supplied correspond to this second method.

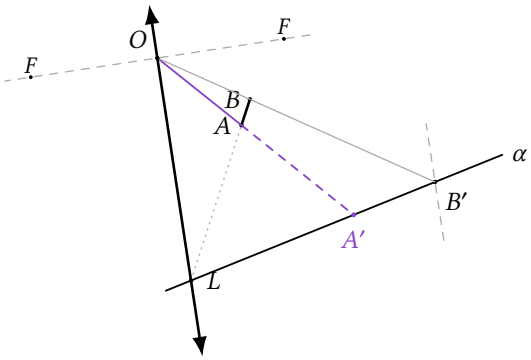


Fig. 36.3

To construct the point A' one can proceed in two ways:

(a) Draw from the point A a ray parallel to POA. After refraction in the lens this ray passes through the focus and intersects the line α at the point A'.

(b) Draw from the point A a ray passing through the optical center O. This ray (in the first case) or its extension (in the second case) intersects the line α precisely at the point A'.

PROBLEM 37

I Put on My Glasses

From the All-Russian Olympiad, regional stage, 2025

Veronika's art
p37 . png

Problem. It often happens that in portrait photographs taken with a smartphone camera (see Fig. 37.1), the reflection in the photographer's glasses shows the image of the smartphone itself. Using the photograph below, determine the radius of curvature of the glasses' reflective surface and estimate the measurement uncertainty. Assume that the distance between the smartphone and the glasses is known exactly and equals $L = 50$ cm. Fig. 37.2 shows the smartphone and the glasses placed next to each other. The lengths needed for the numerical answer in the problem can be measured with a ruler directly on the photographs from the statement.



Fig. 37.1

Solution. Denote the actual (linear) size of the glasses by X and the actual size of the smartphone by Y. Let α be the angular size of the glasses' image on the portrait photo, and β the angular size of the smartphone image on it.

Measure the ratios of the image sizes of the glasses and the smartphone on the two



Fig. 37.2

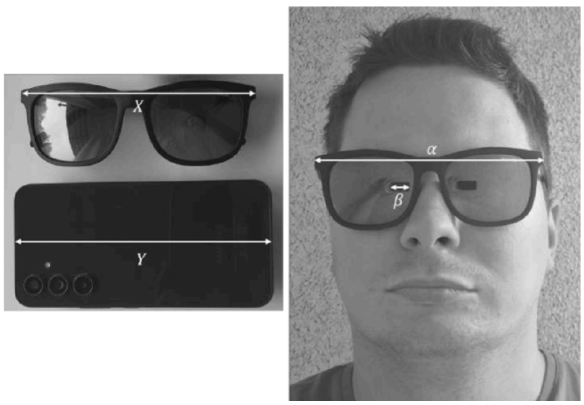


Fig. 37.3

photographs. From the photos we obtain

$$\frac{X}{Y} = (1.1 \pm 0.1), \qquad \frac{\alpha}{\beta} = (12 \pm 1).$$

On the left photograph the ratio of the *linear* sizes of the images coincides with the ratio of the objects' real sizes, while on the right photograph the ratio corresponds to that of the *angular* sizes.

Make a simple sketch of the relative positions of the smartphone image and the glasses on the right photograph. Denote the linear size of the smartphone image by y, and the distance between the smartphone image and the glasses (along the optical axis) by l. In the small-angle approximation the angular sizes of the glasses and the smartphone image on the right photograph are

$$\alpha = \frac{X}{L}, \qquad \beta = \frac{y}{L+l}. \tag{1}$$

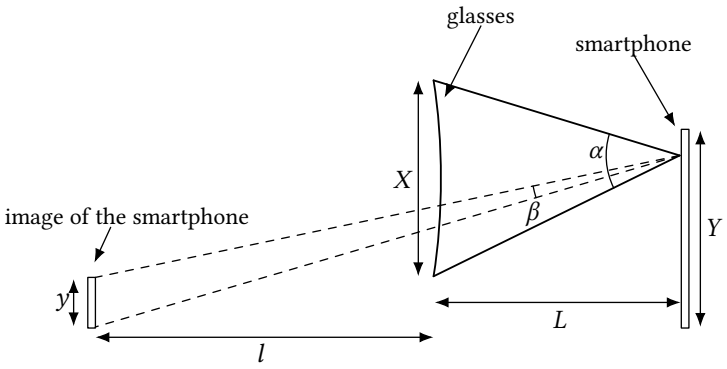


Fig. 37.4

Sketch to be added
ВСТАВИТЬ СХЕМАТИЧЕСКУЮ КАТИНКУ
СО СФЕРИЧЕСКИМ ЗЕРКАЛОМ,
ПОДПИСАТЬ: СХЕМА ПОЛУЧЕНИЯ
ИЗЗОБРАЖЕНИЯ В СФЕРИЧЕСКОМ ЗЕРКАЛЕ

Fig. 37.5

The position of the (virtual) image is determined by the spherical-mirror formula

$$\frac{1}{L} - \frac{1}{l} = -\frac{2}{R}, \tag{2}$$

where R is the radius of curvature of the mirror (the surface of the spectacle lens acting as a mirror). The linear size of the smartphone image is related to its real size by the magnification of the mirror:

$$y = Y\,\frac{l}{L}. \tag{3}$$

Combining Eqs. (1)–(3) we obtain

$$R = \frac{2L}{\dfrac{\alpha}{\beta}\,\dfrac{Y}{X} - 2}. \tag{4}$$

Taking into account the measurement uncertainties, compute the minimal and maximal possible values of the radius of curvature:

$$R_{\min} = \frac{2 \cdot 50\ \text{cm}}{\dfrac{13}{1} - 2} = 9.09\ \text{cm}, \qquad R_{\max} = \frac{2 \cdot 50\ \text{cm}}{\dfrac{11}{1.2} - 2} = 12.24\ \text{cm}.$$

Therefore, the final value of the radius of curvature, including the measurement uncertainty, is

$$\boxed{R = (10.7 \pm 1.6)\ \text{cm}}.$$

PROBLEM 38

The Research Probe

From the All-Russian Olympiad, regional stage, 2025

Veronika's art
p38.png

Problem. A research probe moves along a circular orbit of radius R about the planet *Shelezyaka* and studies the motion of *Boltik*, a small natural satellite of the planet. Boltik's orbit is also circular with radius r ($r < R$) and lies in the same plane as the probe's orbit (see Fig. 38.1). During the observations the probe's instruments recorded that, $t_1 = 165$ min after Boltik crossed the visible edge of the planet's disk, the satellite was at the *maximum angular separation* $\theta_{\max} = 15^\circ$ from the center of Shelezyaka; some time later (greater than t_1) Boltik again crossed the edge of the visible disk of the planet. It is known that between these instants there were no other crossings by Boltik of the planet's visible limb. Shelezyaka is spherical and airless. The mass of Boltik is much smaller than the planet's mass; the probe and the satellite revolve about the planet in the same direction. The angular diameter of the planet as seen from the probe is $2\theta_0 = 6^\circ$. The gravitational constant is $G = 6.67 \cdot 10^{-11}\ \mathrm{N \cdot m^2/kg^2}$.

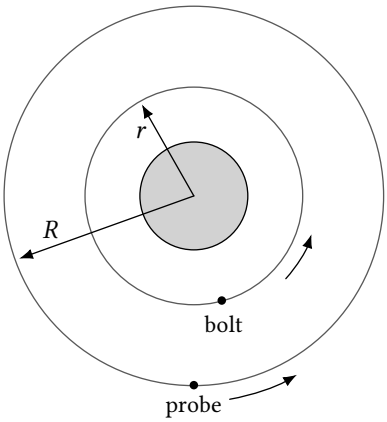


Fig. 38.1

1. Determine the ratio R/r.
2. What is the orbital period T of the probe around Shelezyaka?
3. Find the mean density ρ of Shelezyaka.

Note.

1. The *angular distance* between points A and B is the angle $\angle ASB$, whose sides pass through the points considered and whose vertex S is at the observation point (see Fig. 38.2).
2. The *angular diameter* of an astronomical object is the maximal possible angle α between two tangents to the object's surface, with the vertex at the observation point S (see Fig. 38.3).

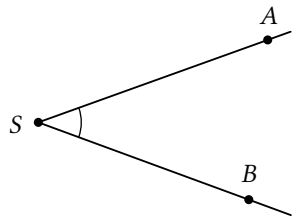


Fig. 38.2

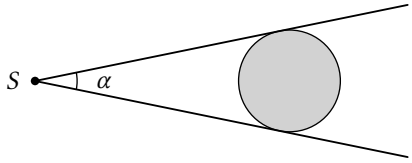


Fig. 38.3

Solution. The angular separation between Boltik and the center of Shelezyaka is maximal when the line "probe–Boltik" is tangent to the satellite's orbit. In this case

$$\sin\theta_{\max} = \frac{r}{R}, \qquad \Rightarrow \qquad \frac{R}{r} = \frac{1}{\sin\theta_{\max}} \approx \frac{1}{\sin 15^\circ} \approx 3.864.$$

Let M be the mass of Shelezyaka. The angular velocities of the satellite and the probe are

$$\omega_s = \sqrt{\frac{GM}{r^3}}, \qquad \omega_p = \sqrt{\frac{GM}{R^3}},$$

hence

$$\omega_s = \omega_p\left(\frac{R}{r}\right)^{3/2} \approx 7.595\,\omega_p.$$

Therefore, as seen from the probe, the satellite makes one revolution around the planet in the time

$$T_{\text{rel}} = \frac{360^\circ}{\omega_s - \omega_p} = \frac{360^\circ}{6.595\,\omega_p} = \frac{T}{6.595}.$$

To find T_{rel}, make the construction shown in the figure: O is the planet's center, S is the probe, and the drawn circle is Boltik's orbit. At the points A or C the satellite lies on the edge of the visible planetary disk; the point B corresponds to the position of *maximum* angular

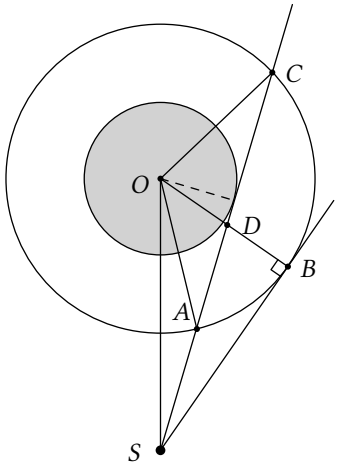


Fig. 38.4

distance between Shelezyaka's center and Boltik (as seen from S). From the geometry of the drawing,

$$\angle ASB = \theta_{\max} - \theta_0 = 12^\circ, \qquad \angle ODA = 90^\circ + \angle ASB = 102^\circ.$$

By the law of sines for triangle SOA,

$$\frac{R}{\sin(\angle OAS)} = \frac{r}{\sin\theta_0},$$

whence, taking into account $\angle OAS > 90^\circ$,

$$\sin(\angle OAS) = \frac{R}{r}\sin\theta_0 = 3.864 \cdot \sin 3^\circ \approx 0.202 \;\Rightarrow\; \angle OAS \approx 168.334^\circ.$$

Find the angle $\angle AOB$:

$$\angle AOB = \angle OAS - \angle ODA = 66.334^\circ.$$

Because the motion is uniform along the circle,

$$\angle AOB = \frac{360^\circ\, t_1}{T_{\text{rel}}} \quad\Rightarrow\quad T_{\text{rel}} = \frac{360^\circ}{66.334^\circ}\, t_1 \approx 5.43 \cdot 165 \text{ min} \approx 895 \text{ min}.$$

With $T = 6.595\, T_{\text{rel}}$ we obtain

$$T \approx 5900 \text{ min}.$$

Let R_{Sh} be Shelezyaka's radius. Then

$$\sin\theta_0 = \frac{R_{\text{Sh}}}{R} \quad\Rightarrow\quad R_{\text{Sh}} = R\sin 3^\circ \approx 0.0523\, R.$$

Writing the planet's mass as $M = \dfrac{4\pi R_{\text{Sh}}^3 \rho}{3}$ and substituting into the expression for the probe's angular velocity on orbit, we get

$$\omega_p^2 = \frac{G \cdot 4\pi R_{\text{Sh}}^3 \rho/3}{R^3} \quad\Longrightarrow\quad \frac{4\pi^2}{T^2} = \frac{4\pi G}{3}\,\rho\,(0.0523)^3,$$

whence

$$\rho = \frac{3\pi}{G\,T^2 \cdot 0.0523^3} \approx 7900\ \text{kg/m}^3.$$

$$\boxed{\frac{R}{r} \approx 3.864, \quad T \approx 5900\ \text{min}, \quad \rho \approx 7.9 \times 10^3\ \text{kg/m}^3}$$

PROBLEM 39

Impulse Acceleration

From the All-Russian Olympiad, regional stage, 2025

Veronika's art
p39.png

Problem. A particle was moving in the plane Oxy in the positive direction of the x-axis with a constant velocity parallel to the x-axis. At time $t = 0$ a time-dependent force lying in the plane Oxy began to act on it. The action of this force produced an acceleration whose direction changed periodically. The magnitude of the acceleration at $t = 0$ was equal to a, and after each time interval 2τ it decreased by a factor of 2. The figure shows the graphs of the time dependence of the components of the particle's acceleration over some initial time interval.

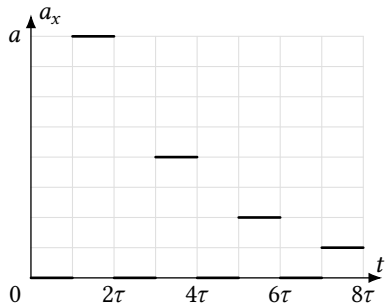


Fig. 39.1

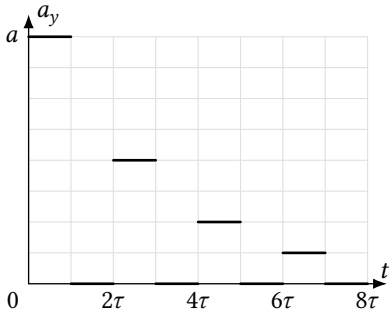


Fig. 39.2

Assuming a and τ are known, determine the maximum and minimum (by magnitude) speeds of the particle if it moves under the action of this force for a long time. During this

motion, the particle's velocity deviated from its initial direction of motion by a maximum angle α.

Note: the sum of an infinite geometric progression

$$b + bq + bq^2 + bq^3 + \cdots = \sum_{i=0}^{\infty} bq^i = \frac{b}{1-q}, \qquad |q| < 1.$$

Solution. We depict, by means of a vector diagram, how the particle's velocity changed with time. One can see that the particle's speed increased monotonically with time; therefore the minimum (by magnitude) speed was at the beginning, and the maximum at the end of the motion (for very large time). Regrouping the vectors of the velocity increments, we obtain (using the formula for the sum of a geometric progression) that the change of the x-projection of the velocity over a long time (an infinite number of cycles) equals $2a\tau$. The y-component changes analogously by $2a\tau$. Hence, the maximum speed in the process is

$$v_\infty = \sqrt{(v_0 + 2a\tau)^2 + (2a\tau)^2}.$$

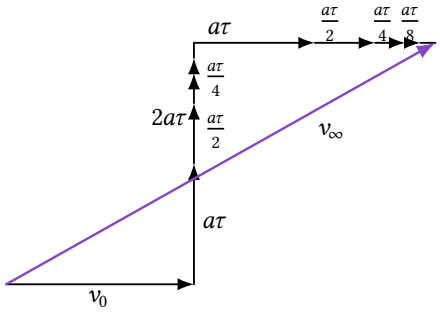


Fig. 39.3

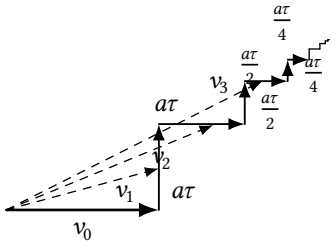


Fig. 39.4

Express the initial speed via $a\tau$. The deflection angle of the velocity vector after time τ from the start satisfies

$$\tan\alpha_1 = \frac{a\tau}{v_0}.$$

After a long time from the start,

$$\tan \alpha_\infty = \frac{2a\tau}{v_0 + 2a\tau}.$$

Two cases are possible:

1. If the initial speed is small, then the maximum deviation of the velocity vector will occur a time τ after the start of the motion:

$$\operatorname{tg} \alpha_1 \geq \operatorname{tg} \alpha_\infty \Rightarrow \alpha_1 = \alpha.$$

The case under consideration is realized for $v_0 \leq 2a\tau$. Then:

$$\operatorname{tg} \alpha = \frac{a\tau}{v_0} \Rightarrow v_{\min} = v_0 = a\tau \cdot \operatorname{ctg} \alpha;$$

$$v_{\max} = v_\infty = \sqrt{(a\tau \cdot \operatorname{ctg} \alpha + 2a\tau)^2 + (2a\tau)^2} = a\tau\sqrt{\operatorname{ctg}^2 \alpha + 4 \operatorname{ctg} \alpha + 8}.$$

2. If $v_0 \geq 2a\tau$, then the maximum deviation of the velocity vector will be in the limiting situation ($\alpha_\infty = \alpha$).

$$\operatorname{tg} \alpha = \frac{2a\tau}{v_0 + 2a\tau} \Rightarrow v_{\min} = v_0 = 2a\tau \cdot (\operatorname{ctg} \alpha - 1);$$

$$v_{\max} = v_\infty = \frac{2a\tau}{\sin \alpha}.$$

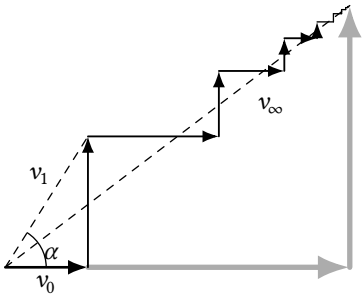


Fig. 39.5

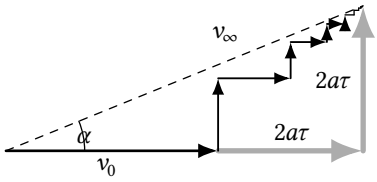


Fig. 39.6

PROBLEM 40

Lord Kelvin's Archive

Veronika's art
p40.png

Problem. In Lord Kelvin's archive a graph of a cyclic process performed on a fixed amount of a monatomic ideal gas was found (see Fig. 40.1). With time, the ink faded and the information about the direction of some processes was lost. The information about what is plotted along the abscissa was also lost. It is only known that one of the following quantities is plotted along the abscissa: volume, pressure, temperature, or density, and the scale is in arbitrary units. The molar heat capacity of the gas C is plotted along the ordinate. Find the maximal possible efficiency η of the cycle.

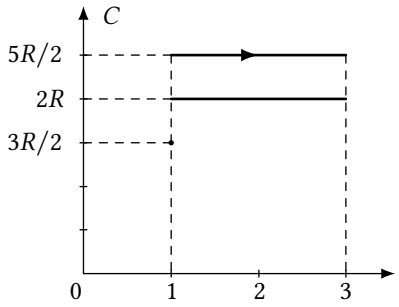


Fig. 40.1

Solution. Note that the cycle consists of three processes with heat capacities $\frac{5}{2}R$, $\frac{3}{2}R$, and $2R$; hence, the first two processes are, respectively, isobaric and isochoric. Let us determine which process corresponds to the heat capacity $2R$.

$$C = \frac{dQ}{dT} = \frac{\frac{3}{2}(P\,dV + V\,dP) + P\,dV}{\frac{1}{R}(P\,dV + V\,dP)} = 2R,$$

from which we obtain $P\,dV = V\,dP$ or $\dfrac{P}{V} = \dfrac{dP}{dV}$, which corresponds to a process where the pressure is proportional to the volume.

Note that on the graph in the statement the isochoric process is shown as a point, which means that the abscissa axis represents either the volume or the density. Consider the case

when the abscissa axis represents the volume. Then, using the fact that one of the processes is an isobaric expansion and that the isochoric process occurs at the smallest value of the volume on the isobar, we obtain the following type of cycle (see Fig. 40.2a).

Find its efficiency η, denoting the minimal pressure and volume by P_0 and V_0:

$$\eta = \frac{A}{Q_{\text{in}}} = \frac{\frac{1}{2} \cdot 2P_0 \cdot 2V_0}{\frac{3}{2}(3P_0 \cdot 3V_0 - P_0V_0) + 3P_0 \cdot 2V_0} = \frac{2P_0V_0}{18P_0V_0} = \frac{1}{9}.$$

If the abscissa of the graph in the statement is ρ (density) or $1/V$, then the corresponding process graph is as in Fig. 40.2b.

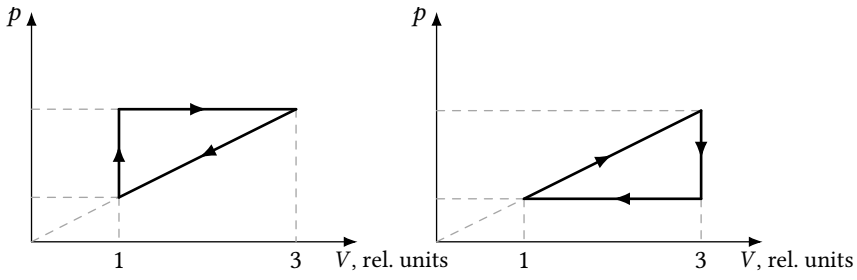


Fig. 40.2

Compute the efficiency of the cycle in this case:

$$\eta = \frac{A}{Q_{\text{in}}} = \frac{\frac{1}{2} \cdot 2P_0 \cdot 2V_0}{\frac{3}{2}(3P_0 \cdot 3V_0 - P_0V_0) + \frac{P_0 + 3P_0}{2} \cdot 2V_0} = \frac{2P_0V_0}{16P_0V_0} = \frac{1}{8}.$$

Thus, the maximal possible efficiency of the cycle is

$$\boxed{\eta_{\max} = \frac{1}{8}}.$$

V

PART V

Problems 41–50

41 Ball on a Spring Inside a Sphere
42 A Luminous Sphere, a Lens in an Aperture, and a Light Cone
43 The “Narrowing” Mirror Cylinder
44 Bulbs…
45 Two Pistons and Two Springs
46 Two Perpendicular Hemispheres
47 The Rod and the Body
48 Hop, Hop, Hop…
49 Diode–Lamp Circuit
50 Linear Oscillations in an Anisotropic Quadratic Potential

PROBLEM 41

Ball on a Spring Inside a Sphere

From the St. Petersburg Physics Olympiad, 2019

Veronika's art
p41.png

Problem. Inside a fixed sphere of radius R one wants to fasten a small ball to a spring in such a way that the ball is located as high as possible (see the figure). The lower end of the spring is hinged at point O. What is the *minimum* length of the spring that should be used? The mass of the ball is m, and its own size is negligible. It is known that the spring is made of such a material that a piece of this spring with natural (unstretched) length R has stiffness k. The mass of the spring is negligible. The acceleration due to gravity is g. There is no friction. The spring does not bend.

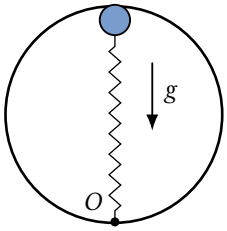


Fig. 41.1

Solution. Denote by L the natural length of the required spring piece. Its stiffness K is easy to compute, knowing that a piece of the same spring of length R has known stiffness k. Indeed, for a given spring material and wire thickness, the stiffness is inversely proportional to the length. This is easy to understand if, for example, one notes that two identical springs connected in series stretch twice as much under the weight of a given load as each of these springs separately. Therefore,

$$K = \frac{kR}{L}. \tag{1}$$

The extension of such a spring placed in our system is, at first glance, $\Delta x = L - 2R$. At first one might think that one should take such a spring that the spring force compensates the weight, $K\Delta x \geq mg$.

However, it is easy to show that this position is unstable with respect to an accidental rotation of the spring about the hinge O. For stability, a longer spring is needed, as we shall now see.

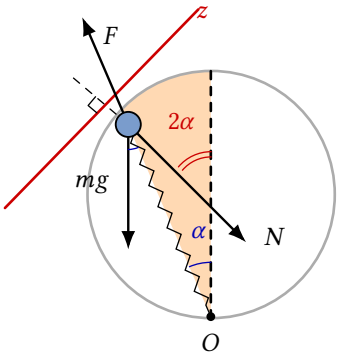


Fig. 41.2

Indeed, let the spring deviate from the vertical by a small angle α. The forces acting on the ball in this case are shown in the figure (F is the spring force, N is the reaction force of the sphere). The figure also shows the necessary relations between the angles.

For the ball to return to the vertical position, it is necessary that the z-projection of the resultant of the forces N, F and mg be positive, i.e.

$$F \sin\alpha \geq mg \sin(2\alpha).$$

Here the force F is determined by the length of the spring, which equals $2R\cos\alpha$ (from the right triangle in the figure), hence the extension of the spring is

$$L - 2R\cos\alpha,$$

and the inequality can be rewritten as

$$K(L - 2R\cos\alpha)\ \sin\alpha \geq mg \sin(2\alpha).$$

It is sufficient that this inequality hold already for small deviations α, i.e. when $\sin\alpha \simeq \alpha$, $\cos\alpha \simeq 1$, and $\sin 2\alpha \simeq 2\alpha$. Therefore,

$$K(L - 2R)\alpha \geq 2mg\,\alpha \qquad \Rightarrow \qquad K(L - 2R) \geq 2mg.$$

Substituting (1) and solving for L, we obtain the answer

$$\boxed{L = \frac{2kR^2}{kR - 2mg}}.$$

PROBLEM 42

A Luminous Sphere, a Lens in an Aperture, and a Light Cone

From the St. Petersburg Physics Olympiad, 2022

Veronika's art
p42.png

Problem. An opaque screen has a circular hole of radius R. A thin converging lens with known focal length F is inserted into the opening. On the optical axis of the lens, at a distance l from its center, there is the center of a luminous sphere (see the figure; $l > F$, the sphere does not intersect the focal plane of the lens). It is found that, on the other side of the screen, at some distance from the lens, the rays emitted by the sphere fill a cone with aperture angle 2α. Find the radius r of the luminous sphere. Assume the radius of the sphere is smaller than R.

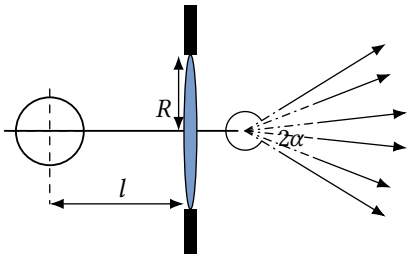


Fig. 42.1

Solution. Any ray that passes through the lens originates from some point on the surface of the sphere. Since the problem has an axis of symmetry—the principal optical axis of the lens—all constructions will be carried out in the plane of the drawing.

Draw tangents AK and $A'K'$ from the edges of the lens to the sphere. Obviously, only the light emitted by points of the arc ABA' will hit the lens (see Fig. 42.2). The cone is formed by the rays that deviate the most from the principal optical axis after passing through the lens.

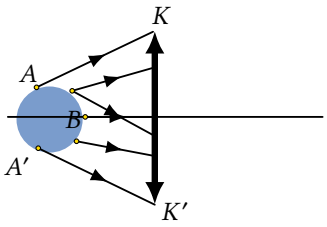


Fig. 42.2

First method

Choose an arbitrary point C on the arc AB. Behind the lens all rays emerging from C will pass through the image of point C—point C' (see Fig. 42.3). From the construction it is clear that, behind the lens, the strongest deviation from the principal optical axis is produced by the ray CK (shown in green).

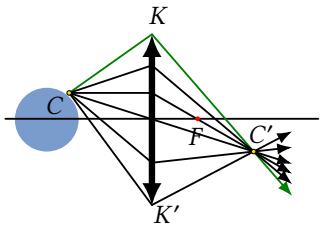


Fig. 42.3

Now consider the rays that strike the lens edge K from different points of the sphere surface—points A, C, D (see Fig. 42.4; for clarity we draw the rays AK, CK, DK in different colors). To construct the paths of these rays behind the lens, one needs, for each of them, to construct the "secondary" focus in the focal plane—the point where the lens gathers the rays that are parallel to the given ray. It is easiest to construct the secondary focus using the fact that rays passing through the lens center are not refracted. Corresponding constructions are performed with matching colors.

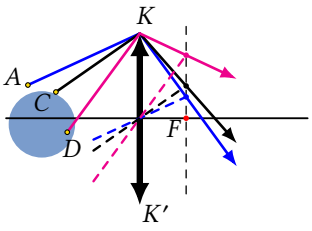


Fig. 42.4

From the constructions it is clear that behind the lens, the ray that makes the largest angle with the principal optical axis is the ray that enters point K at the smallest angle to the principal optical axis, that is, ray AK, highlighted in blue.

Thus, the generatrix of the cone is determined by ray AK after it is refracted by the lens.

Let us consider ray AK more carefully (see Fig. 42.5). Denote by φ the angle between it and the principal optical axis (marked in red). We have proved that after the lens this ray goes at angle α, which is known from the problem statement.

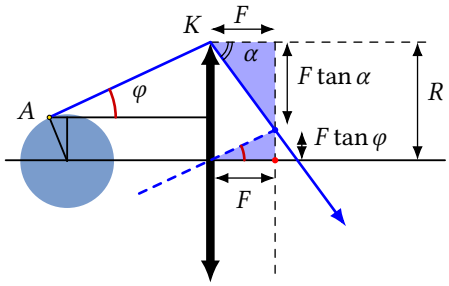


Fig. 42.5

Consider the two right triangles marked in blue in the figure. Each of them has a horizontal leg of length F; one of these triangles has angle α, the other φ. As can be seen from the figure, the sum of the two vertical legs of these triangles is equal to the lens radius, therefore

$$R = F\tan\alpha + F\tan\varphi \qquad \Rightarrow \qquad \tan\varphi = \frac{R}{F} - \tan\alpha. \tag{1}$$

So, we now know φ.

Continue tracing the ray AK, now to the left of the lens. Consider the two right triangles—the green and the orange ones (see Fig. 42.6), which share the common vertex A. The sum of the vertical legs of these two triangles is also equal to R—this condition will allow us to find the answer.

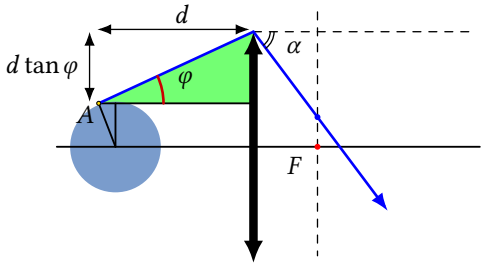


Fig. 42.6

The orange triangle has hypotenuse r—the desired sphere radius. The angle between the hypotenuse and the vertical leg in it equals φ, because the hypotenuse is perpendicular to the ray AK considered. Therefore the vertical leg of the orange triangle equals $r\cos\varphi$.

The horizontal leg of the green triangle is the sum $d = l + r\sin\varphi$, where l is the known distance from the sphere center to the lens center, and $r\sin\varphi$ is the horizontal leg of the orange triangle. Hence the vertical leg of the green triangle equals $d\tan\varphi = (l + r\sin\varphi)\tan\varphi$.

Summing the vertical legs of the green and orange triangles, write the equation that allows one to find r:

$$R = r\cos\varphi + (l + r\sin\varphi)\tan\varphi.$$

Solve for r:

$$r = \frac{R - l\tan\varphi}{\sin\varphi\tan\varphi + \cos\varphi} = \big(R - l\tan\varphi\big)\cos\varphi. \tag{2}$$

Finally, using the trigonometric identity

$$\cos^2\varphi = \frac{1}{1 + \tan^2\varphi}$$

and formula (1) for $\tan\varphi$, we obtain from (2) the answer

$$r = \frac{R - l\left(\frac{R}{F} - \tan\alpha\right)}{\sqrt{1 + \left(\frac{R}{F} - \tan\alpha\right)^2}}.$$

Second method

Let the sought radius of the sphere be r. Consider an arbitrary point C on the sphere at height h and at distance d from the lens center (see Fig. 42.7).

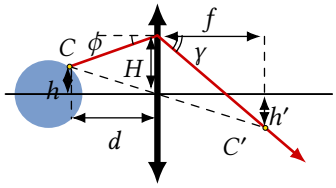


Fig. 42.7

Suppose that some ray leaves this point and reaches the lens at height H. After the lens, this ray goes into the image of point C, which we denote by C', assuming that it is located below the principal optical axis by h' and at distance f from the center of the lens. Denote by ϕ the angle at which this ray travels before the lens, and by γ the angle after the lens.

Then, from the thin-lens formula,

$$\frac{1}{d} + \frac{1}{f} = \frac{1}{F}, \qquad \frac{h}{d} = \frac{h'}{f}.$$

For the angles ϕ and γ, we have

$$\tan\phi = \frac{H - h}{d}, \qquad \tan\gamma = \frac{H + h'}{f}.$$

From this one can obtain

$$\tan\gamma = \frac{H}{F} - \tan\phi.$$

From this it is clear that, in order to increase the angle γ, it is advantageous to increase H and decrease ϕ. Thus, we arrive at the conclusion that ray AK, introduced in the first

solution, after being refracted by the lens, will have the greatest inclination to the principal optical axis.

Therefore, for ray AK, one should set $H = R$, $\gamma = \alpha$. This makes it possible to find the angle φ between AK and the principal optical axis from the previous relation — since for ray AK, $\varphi = \phi$:

$$\tan\varphi = \frac{R}{F} - \tan\alpha.$$

It remains to substitute into the first of the relations above

$$H = R, \qquad h = r\cos\varphi, \qquad d = l + r\sin\varphi,$$

as we did when solving the problem by the first method, as well as the angle φ found above:

$$\tan\varphi = \frac{R - r\cos\varphi}{l + r\sin\varphi}, \qquad \text{where } \tan\varphi = \frac{R}{F} - \tan\alpha,$$

and solve for r.

$$\boxed{r = \frac{R - l\left(\dfrac{R}{F} - \tan\alpha\right)}{\sqrt{1 + \left(\dfrac{R}{F} - \tan\alpha\right)^2}}}$$

PROBLEM 43

The "Narrowing" Mirror Cylinder

From the St. Petersburg Physics Olympiad, 2023

Veronika's art
p43.png

Problem. A mirror (specular) cylinder is mounted vertically and fixed to a wall flush with it. On the wall behind the cylinder a long bright horizontal strip shines (Fig. 43.1a: the strip is sketched as a gray band). A photographer stands opposite the cylinder and photographs it against the background of the strip and the wall. In the photograph the cylinder appears visually narrower at the level of the bright strip (Fig. 43.1b). When the lamp is switched off, the effect disappears. Why does this happen?

Assume the full angular width of the cylinder is $2\alpha_0$. Find the apparent angular width of the cylinder in the region of maximum optical distortion. The cylinder axis is in the middle of the frame; the image plane is parallel to the wall. (Remark: the angular size of an object is the angle under which it is seen, cf. Fig. 43.1c, where φ is the angular size.)

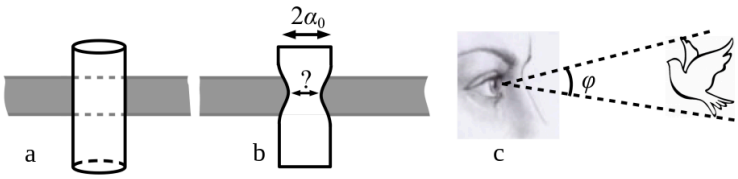


Fig. 43.1

Solution. Let x be the distance between the observer's eye N and the cylinder axis. It is easy to relate x to the cylinder's angular half-size α_0 for the observer (Fig. 43.2, the picture is drawn in the horizontal plane):

$$\sin\alpha_0 = \frac{R}{x} \quad \Rightarrow \quad x = \frac{R}{\sin\alpha_0}. \tag{1}$$

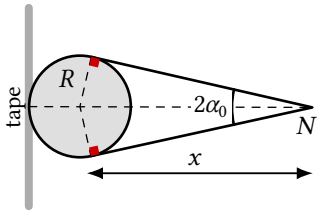


Fig. 43.2

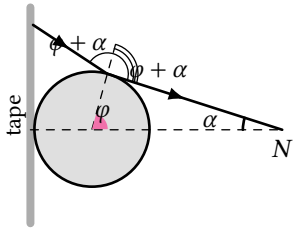


Fig. 43.3

Now consider a ray that lies in the horizontal plane, emanates from the bright strip, reflects from the cylinder, and reaches the observer's eye at an angle α to the perpendicular from the eye to the wall (see Fig. 43.3).

Such rays will strike different parts of the cylinder, corresponding to different angles φ in the figure. The angle of incidence and the angle of reflection will then both be equal to $\varphi + \alpha$.

The ray that hits the cylinder at the smallest angle φ is the one coming from a very distant part of the strip. It is exactly this ray that is shown in Fig. 43.4, and it is exactly this ray that determines which part of the cylinder will be illuminated. The non-illuminated part of the cylinder consists of the points lying inside the angle φ highlighted in Fig. 43.4, as well as the points of the cylinder symmetric to them with respect to NO.

From the same figure it is clear that the angles $\varphi + \alpha$ and φ together form a right angle. Therefore,

$$2\varphi + \alpha = 90^\circ. \tag{2}$$

According to the statement, we must find 2α, and now it is clear that this is elementary once we determine φ.

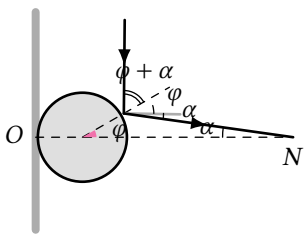


Fig. 43.4

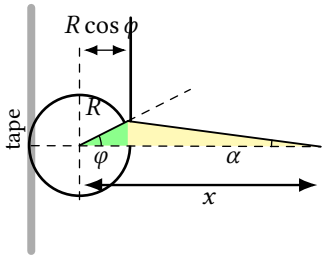


Fig. 43.5

It remains to consider the green and yellow triangles in Fig. 43.5, writing for their common leg

$$R\sin\varphi = (x - R\cos\varphi)\operatorname{tg}\alpha.$$

Let us transform:

$$\begin{aligned} R\sin\varphi\cos\alpha &= (x - R\cos\varphi)\sin\alpha \\ \Rightarrow\ R(\sin\varphi\cos\alpha + \cos\varphi\sin\alpha) &= x\sin\alpha \\ \Rightarrow\ R\sin(\varphi+\alpha) &= x\sin\alpha. \end{aligned}$$

Next we use that, according to (2), $\alpha = 90^\circ - 2\varphi$:

$$R\sin(90^\circ - \varphi) = x\sin(90^\circ - 2\varphi).$$

Hence

$$R\cos\varphi = x\cos 2\varphi = x(2\cos^2\varphi - 1).$$

.

This is a quadratic equation in $\cos\varphi$:

$$2x\cos^2\varphi - R\cos\varphi - x = 0 \qquad \Rightarrow \qquad \cos\varphi = \frac{R \pm \sqrt{R^2 + 8x^2}}{4x}.$$

Substitute x from (1), discard the nonphysical root, and obtain the result. The apparent narrowing (the bright waist) corresponds to the angular size

$$2\alpha = 180^\circ - 4\varphi, \quad \text{where} \quad \varphi = \arccos\left(\frac{\sin\alpha_0 + \sqrt{\sin^2\alpha_0 + 8}}{4}\right).$$

PROBLEM 44

Bulbs...

From the St. Petersburg Physics Olympiad, 2021

Veronika's art
p44.png

Problem. There are N identical bulbs filled with equal amounts of an ideal gas. The temperature of each bulb is kept constant; the given values are T_1, T_2, ..., T_N. All bulbs are connected in a chain by thin capillary tubes and the chain is attached to a manometer M (see the figure). The manometer reads a pressure P_0.

The experimenter wants the manometer to read a new value P. To achieve this, he connects one more identical bulb to the chain, also filled with the same gas. What temperature T_{N+1} must be maintained in this additional bulb?

You may neglect the total volume of the connecting tubes and the gas contained in the manometer.

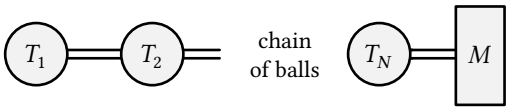


Fig. 44.1

Solution. Let ν be the initial amount of substance of the gas in each bulb, and V the volume of each bulb.

After the bulbs are connected with tubes, the gas starts to redistribute between bulbs until the pressures in all bulbs become the same and equal to the manometer reading P_0. Denote by ν_i the amount of substance of gas in the i-th bulb when equilibrium is established in this first situation. Then, in each bulb, the ideal-gas (Clapeyron–Mendeleev) equation gives

$$P_0 = \frac{\nu_i R T_i}{V}. \tag{1}$$

On the other hand, the total amount of substance of gas in the system equals $N\nu$, hence

$$\sum_{i=1}^{N} \nu_i = N\nu. \tag{2}$$

Expressing ν_i from (1) as $\nu_i = \dfrac{P_0 V}{RT_i}$ and substituting into (2), we obtain

$$\sum_{i=1}^{N} \frac{P_0 V}{RT_i} \;=\; N\nu. \tag{3}$$

When one more bulb is attached to the system, by the same reasoning we get the relation

$$\frac{PV}{R}\sum_{i=1}^{N+1}\frac{1}{T_i} \;=\; (N+1)\nu \qquad \text{or} \qquad \frac{P}{R}\left(\sum_{i=1}^{N}\frac{1}{T_i}+\frac{1}{T_{N+1}}\right) \;=\; (N+1)\frac{\nu}{V}. \tag{4}$$

From (3) we express ν/V and substitute into (4):

$$\frac{\nu}{V} \;=\; \frac{P_0}{NR}\sum_{i=1}^{N}\frac{1}{T_i}, \qquad \Rightarrow \qquad \frac{P}{R}\left(\sum_{i=1}^{N}\frac{1}{T_i}+\frac{1}{T_{N+1}}\right) = (N+1)\frac{\nu}{V} \;= (N+1)\frac{P_0}{NR}\sum_{i=1}^{N}\frac{1}{T_i}. \tag{5}$$

Hence we readily isolate the desired temperature:

$$\frac{P}{R}\sum_{i=1}^{N}\frac{1}{T_i}+\frac{P}{R}\frac{1}{T_{N+1}} \;=\; \frac{(N+1)P_0}{NR}\sum_{i=1}^{N}\frac{1}{T_i} \;\Rightarrow\; \frac{P}{R}\frac{1}{T_{N+1}} \;=\; \left(\frac{(N+1)P_0}{NR}-\frac{P}{R}\right)\sum_{i=1}^{N}\frac{1}{T_i} \;\Rightarrow (1)$$

$$\boxed{T_{N+1} \;=\; \frac{NP}{(N+1)P_0 - NP}\left(\sum_{i=1}^{N}\frac{1}{T_i}\right)^{-1}}. \tag{6}$$

Important. The denominator must neither turn to zero nor become negative. Therefore, if $(N+1)P_0 \le NP$, no solution exists.

Answer. If $(N+1)P_0 > NP$, the required temperature is given by formula (6); otherwise there is no solution.

Remark. The problem admits an analogy with a parallel connection of N conductors with different resistances: the role of current through each branch is played by ν_i, the role of resistances—by T_i, and the "voltage" corresponds to PV/R. Equation (2) mirrors current summation in parallel circuits, while (1) is analogous to the law of equal voltages across parallel branches. It is thus not surprising that the final result reproduces a law akin to computing the equivalent resistance of parallel conductors.

PROBLEM 45

Two Pistons and Two Springs

From the St. Petersburg Physics Olympiad, 2017

Veronika's art
p45 . png

Problem. Inside a vessel there are two pistons of mass m and two springs of stiffness k. One spring connects the pistons to each other, the other connects the lower piston with the bottom of the vessel (see the figure). The space below the pistons is filled with an ideal gas, and initially the system is in equilibrium; both springs are undeformed and have length L. A small through hole O is then opened in the lower piston. At what heights will the pistons be located when the system reaches a new equilibrium? Assume the temperature in the end is equal to the initial one. Neglect atmospheric pressure, the weight of the gas, and friction. The acceleration due to gravity is g.

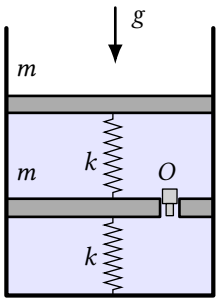


Fig. 45.1

Solution. At the initial moment the springs are undeformed, therefore they do not act on the pistons. Hence the weight of each piston is balanced only by gas pressures. From equilibrium of the upper piston we obtain that the pressure of the gas beneath it (in the upper compartment) equals

$$p_0 = \frac{mg}{S},$$

where S is the piston area. On the lower piston the pressure force from the upper compartment acts from above, $p_0 S = mg$, and from below there is the pressure of the gas in the lower compartment. The difference of these forces balances the weight mg of the lower piston, therefore the pressure of the gas in the lower compartment is initially $2p_0$.

Denote by p the gas pressure in both compartments after the final equilibrium is established. Suppose that after equilibrium the upper spring is compressed by x_1, and the lower

one by x_2. (If in fact a spring is stretched rather than compressed, this corresponds to a negative value of x_1 or x_2.) With these notations the final volumes of the gas compartments are

$$V_1 = (L - x_1)S, \qquad V_2 = (L - x_2)S.$$

Using the Clapeyron–Mendeleev (ideal gas) equation for the amounts of substance in the compartments at the beginning, ν_1 and ν_2, namely

$$\nu_1 = \frac{p_0 LS}{RT}, \qquad \nu_2 = \frac{2p_0 LS}{RT},$$

and noting that the total amount of substance $\nu_1 + \nu_2 = \dfrac{3p_0 LS}{RT}$ is found in the volume $V_1 + V_2$ at pressure p (the temperature T did not change), we get

$$\nu_1 + \nu_2 = \frac{p(V_1 + V_2)}{RT} \quad \Rightarrow \quad \frac{3mg}{S} L = p(2L - x_1 - x_2). \tag{1}$$

In the final equilibrium, the forces acting on each piston are again balanced. For the upper piston this gives

$$mg = pS + kx_1. \tag{2}$$

For the lower piston the gas pressure p is the same above and below; therefore its weight is balanced by the difference of the spring tensions:

$$mg + kx_1 = kx_2. \tag{3}$$

We solve the system (1)–(3). It is convenient to eliminate many dimensional parameters by introducing the relative compressions of the springs

$$\alpha_{1,2} = x_{1,2}/L,$$

which also allows us to cancel L in (1). In the remaining two equations, switching to $\alpha_{1,2}$ highlights the dimensionless parameter

$$a = \frac{mg}{kL},$$

and the system (1)–(3) becomes

$$3a = \frac{pS}{kL}(2 - \alpha_1 - \alpha_2), \qquad a = \frac{pS}{kL} + \alpha_1, \qquad a + \alpha_1 = \alpha_2. \tag{4}$$

Expressing from the second equation $\dfrac{pS}{kL} = a - \alpha_1$ and substituting it into the first one together with α_2 from the third, we obtain a quadratic equation for α_1:

$$3a = (a - \alpha_1)(2 - a - 2\alpha_1).$$

Expanding the brackets,

$$2\alpha_1^2 - (a + 2)\alpha_1 - a(a + 1) = 0,$$

so the two solutions for the upper spring are

$$\alpha_1 = \frac{a + 2 \pm \sqrt{(a + 2)^2 + 8a(a + 1)}}{4} = \frac{a + 2 \pm (3a + 2)}{4}.$$

It is easy to see that taking the "plus" root gives $\alpha_1 = 1 + a$. But $a > 0$, and the relative compression α_1 cannot exceed 1 (as the usual compression x_1 cannot exceed the length L of an undeformed spring). Hence such a solution is non-physical (one can also verify that it would lead to a negative p). The "minus" root yields a physically acceptable answer $\alpha_1 = -a/2$, $\alpha_2 = a/2$.

Thus, it would seem we have the result: the upper spring is stretched by $x_1 = L\alpha_1 = mg/(2k)$, while the lower one is compressed by the same amount. However, note that the lower spring's relative compression $\alpha_2 = a/2$ must also not exceed 1, and therefore the obtained solution is non-physical for $a > 2$.

In that case the lower piston simply lies on the bottom of the vessel, because the spring stiffness is insufficient to support it. Equation (3) no longer holds; instead one must set $x_2 = L$ (i.e. $\alpha_2 = 1$). Instead of the system (4) we now have

$$3a = \frac{pS}{kL}(1 - \alpha_1), \qquad a = \frac{pS}{kL} + \alpha_1.$$

From the second equation express $\dfrac{pS}{kL}$, substitute into the first, and solve the resulting quadratic equation

$$3a = (a - \alpha_1)(1 - \alpha_1)$$

with respect to α_1:

$$\alpha_1^2 - (a+1)\alpha_1 - 2a = 0 \quad \Rightarrow \quad \alpha_1 = \frac{a + 1 \pm \sqrt{(a+1)^2 + 8a}}{2}.$$

The "plus" root must again be discarded since it is definitely greater than one. The upper piston then is located above the bottom at a height $L - L\alpha_1$.

Answer.

- If $\dfrac{mg}{kL} \le 2$, the upper piston does not move, and the lower one goes down by $\dfrac{mg}{2k}$.
- If $\dfrac{mg}{kL} > 2$, the lower piston rests on the bottom, while the upper piston is located at the height

$$L\frac{\sqrt{a^2 + 10a + 1} + 1 - a}{2}, \qquad a = \frac{mg}{kL}.$$

PROBLEM 46

Two Perpendicular Hemispheres

From the Rosatom Olympiad, 2025

Veronika's art
p46 . png

Problem. A dielectric sphere of radius R, uniformly charged with total charge q, is cut into two identical parts. A second dielectric sphere of radius $2R$, uniformly charged with total charge $3q$, is also cut into two identical parts (see the left part of the figure). Then one half of the smaller sphere and one half of the larger sphere are arranged and held as shown in the right sketch: the two hemispheres share the same center, and their cutting planes are mutually perpendicular. Find the magnitude and direction of the force of interaction between the hemispheres.

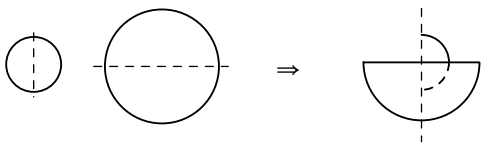
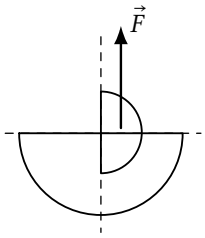


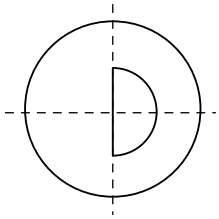
Fig. 46.1

Solution. Let us prove that the interaction force between two uniformly charged concentric hemispheres whose base planes are perpendicular to each other is perpendicular to the base plane of the larger hemisphere (as shown in Fig. 46.2).



$\vec{F}$ is perpendicular to the base of the large hemisphere

Fig. 46.2

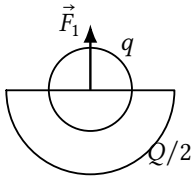


The interaction force is zero: $F = 0$

Fig. 46.3

Indeed, suppose that inside a uniformly charged sphere there is a uniformly charged hemisphere. Clearly, the force of their mutual interaction is zero, since all charges of the smaller hemisphere lie inside the larger sphere, which exerts no force on any point charge placed inside it (Fig. 46.3). This means that the forces acting on the smaller hemisphere from the lower and the upper hemispheres of the large sphere are equal in magnitude and opposite in direction. On the other hand, these two forces must be obtained from each other by reflection with respect to the base plane of the large hemisphere, since the two charge systems are obtained from each other by such a reflection (see Fig. 46.4). But if these forces were not directed at a right angle to this plane, they would not add up to zero. Hence the stated claim about the direction of the force exerted by the large hemisphere on the small one (Fig. 46.2) follows.

This means that if we found the force F_1 of interaction between the large hemisphere of radius $2R$ with charge $Q/2$ and the sphere of radius R with charge q, then the required interaction force F of the hemispheres would be exactly one half of F_1 (Fig. 46.4).



The required force is $\frac{1}{2}$ of $\vec{F}_1$

Fig. 46.4

Let us find the force F_1.

To this end, note that the electric field of a uniformly charged sphere outside it is the same as the field of a point charge located at its center. This means that the small sphere of radius R, carrying charge q, acts on the concentric hemisphere of radius $2R$ with charge $Q/2$ in the same way as a point charge q placed at the center of the hemisphere acts on the hemisphere. By Newton's third law, the force exerted by the hemisphere on the point charge is the same (see Fig. 46.5). Thus, the computation of F_1 reduces to finding the interaction force between the hemisphere and a point charge at its center.

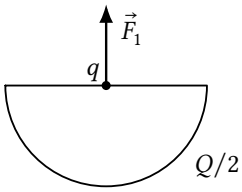


Fig. 46.5

We can compute this force as follows. Mentally divide the hemisphere into small surface elements, find the force exerted on the point charge by each element, and sum the force vectors. Moreover, one may sum not the vectors themselves but their projections on the axis of symmetry of the problem (shown by the dashed line in Fig. 46.6), since by symmetry the force can be directed only along this axis.

From a surface element of area ΔS the point charge experiences a force whose projection on the symmetry axis is

$$\Delta F \cos\alpha = k \frac{q\,\Delta Q}{(2R)^2} \cos\alpha,$$

where ΔQ is the charge of the element, α is the angle between the force vector and the symmetry axis, k is Coulomb's constant, and $2R$ is the hemisphere radius (see Fig. 46.6).

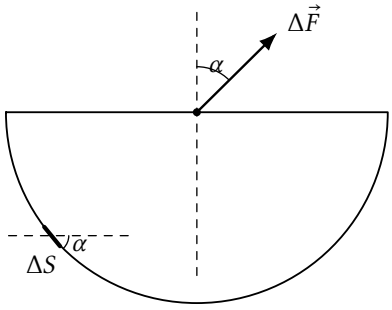


Fig. 46.6

Obviously, the charge of the element is proportional to its area:

$$\Delta Q = \frac{Q/2}{2\pi(2R)^2}\,\Delta S = \frac{Q}{4\pi(2R)^2}\,\Delta S$$

($Q/2$ is the charge of the hemisphere). Since the angle between the element and a line parallel to the base plane of the hemisphere equals the angle between the force $\Delta\vec{F}$ and the symmetry axis (Fig. 46.6), the product $\Delta S \cos\alpha$ is the projection of the area of the considered element onto the base plane of the hemisphere. Therefore, summing the forces from all elements into which the hemisphere can be divided gives for the point charge:

$$F_1 = \sum_i \Delta F_i \cos\alpha_i = k\,\frac{q}{(2R)^2}\,\frac{Q}{4\pi(2R)^2} \sum_i \Delta S_i \cos\alpha_i,$$

and the sum $\sum_i \Delta S_i \cos\alpha_i$ equals the base area of the hemisphere, $\pi(2R)^2$. As a result we

obtain, for the interaction force between the point charge q and the hemisphere of radius $2R$ carrying charge $Q/2$,

$$F_1 = k\,\frac{qQ}{4(2R)^2} = k\,\frac{qQ}{16R^2}.$$

Hence, finally: the force exerted by the hemisphere of radius $2R$ carrying charge $Q/2$ on the concentric hemisphere of radius R carrying charge $q/2$ with a base plane perpendicular to the base plane of the larger hemisphere is directed perpendicular to the base plane of the larger hemisphere and has magnitude

$$F = \frac{1}{2}F_1 = k\,\frac{qQ}{32R^2}.$$

Since $Q = 3q$, the answer is

$$F = k\,\frac{3q^2}{32R^2}.$$

PROBLEM 47

The Rod and the Body

From the Rosatom Olympiad, 2019

Veronika's art
p47 . png

Problem. A very small body of mass m is attached to the end of a weightless rod. The other end of the rod is hinged to a horizontal surface. If the rod is initially set at some angle to the vertical and then released, it falls onto the surface in time t. What time will a similar rod take to fall onto the surface if a small body of mass $2m$ is attached at its midpoint, the rod is set at the same initial angle to the vertical, and released?

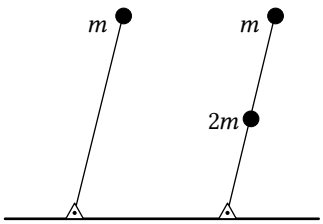


Fig. 47.1

Solution. Let us compare the angular velocities of the rods at the moment when they are inclined at the same angle. First, consider the first rod (with one body). When it is at an angle β to the vertical, the decrease in potential energy is

$$\Delta E_p = mgl\big(\cos\alpha - \cos\beta\big),$$

where α is the initial angle between the rod and the vertical (see the figure). Therefore, the law of conservation of mechanical energy gives

$$mgl\big(\cos\alpha - \cos\beta\big) = \frac{mv^2}{2} = \frac{m\omega^2 l^2}{2},$$

where v and ω are the speed of the body and the angular speed of the rod at the moment when it is inclined at an angle β to the vertical. Hence we find

$$\omega = \sqrt{\frac{2g(\cos\alpha - \cos\beta)}{l}}.$$

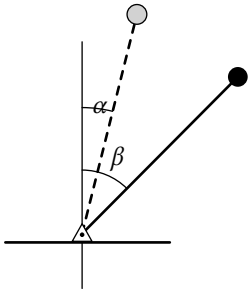


Fig. 47.2

Now consider the second rod at the moment when it is inclined at an angle β to the vertical (see the figure). The decrease in potential energy for it is determined by

$$\Delta E_p = mgl\big(\cos\alpha - \cos\beta\big) + 2mg\frac{l}{2}\big(\cos\alpha - \cos\beta\big) = 2mgl\big(\cos\alpha - \cos\beta\big).$$

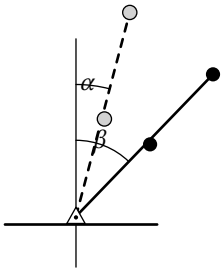


Fig. 47.3

The law of conservation of mechanical energy for this rod gives

$$2mgl\big(\cos\alpha - \cos\beta\big) = \frac{m\omega_1^2 l^2}{2} + \frac{2m\omega_1^2 (l/2)^2}{2} = \frac{3m\omega_1^2 l^2}{4},$$

where ω_1 is the angular speed of the second rod at the moment when it is inclined at an angle β to the vertical. Hence we find

$$\omega_1 = \sqrt{\frac{8g(\cos\alpha - \cos\beta)}{3l}}.$$

It follows that the ratio of the times required for the rods to execute each small rotation ($\Delta\varphi = \omega\Delta t$) is

$$\frac{\Delta t}{\Delta t_1} = \frac{\omega_1}{\omega} = \sqrt{\frac{4}{3}}.$$

And this does not depend on the angle β. This means that the ratio of the total times of motion is the same. That is,

$$t_1 = \sqrt{\frac{3}{4}}\,t.$$

PROBLEM 48

Hop, Hop, Hop...

From the All-Siberian Olympiad, 2018

Veronika's art
p48.png

Problem. A ball is thrown onto a horizontal surface that has a fine triangular corrugation. The launch is at an angle α to the horizontal. Before the first impact with the surface the maximal height of the ball's trajectory equals H (see the figure). The ball elastically rebounds 11 times from the inclined faces of the triangles and returns to the starting point along the same path. The ball hits only the inclined faces of the surface. Determine the greatest height of the ball's trajectory $H_{\max}$.

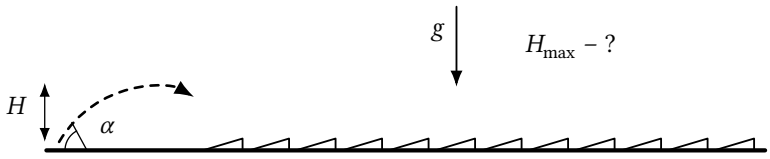


Fig. 48.1

Solution. Let β be the inclination angle of the corrugated (triangular) surface with respect to the horizontal, and let U_0 be the ball's speed at launch (its maximal speed).

After the k-th impact, the reflection angle (measured from the horizontal) changes with respect to the angle of incidence as

$$\alpha_k = \alpha_{k-1} + 2\beta.$$

For the ball to return to its original trajectory, during the first five impacts it must rebound forward at angles $\alpha_k < \pi/2$, while at the sixth impact it must strike the surface perpendicularly. Hence

$$\alpha + 10\beta = \frac{\pi}{2} - \beta, \qquad \Rightarrow \qquad \beta = \frac{\pi - 2\alpha}{22}.$$

The height of the trajectory (the maximal altitude on that arc) after the k-th bounce is

$$H_k = \frac{U_0^2 \sin^2 \alpha_k}{2g},$$

and the initial height is

$$H = \frac{U_0^2 \sin^2 \alpha}{2g}.$$

The height is maximal on the rightmost segment, therefore

$$H_{\max} = \frac{U_0^2 \sin^2\left(\frac{\pi}{2} - \beta\right)}{2g} = H \frac{\sin^2\left(\frac{\pi}{2} - \beta\right)}{\sin^2 \alpha}.$$

Using $\beta = \dfrac{\pi - 2\alpha}{22}$, we obtain

$$\boxed{H_{\max} = H \frac{\sin^2\left(\dfrac{10\pi + 2\alpha}{22}\right)}{\sin^2 \alpha}}.$$

PROBLEM 49

Diode–Lamp Circuit

From the olympiad "Pokori Vorob'evy gory!", 2022

Veronika's art
p49.png

Question. What is the current flowing through a diode whose I–V characteristic is shown in the right-hand graph of the problem if it is connected to a source with emf $\mathscr{E} = 4$ V and internal resistance $r = 4\ \Omega$?

Problem. In the figure on the left a circuit with a diode is shown; the diode's I–V characteristic in the forward (conducting) state is given in the right-hand graph. The lamp has the I–V characteristic

$$I = I_0 \cdot \sqrt{\frac{U}{\mathscr{E}}},$$

where $I_0 = 0.5$ A. The internal resistances of both sources are negligibly small, the emf is $\mathscr{E} = 5$ V, and the resistor has resistance $R = 200\ \Omega$. Find the power consumed by the diode.

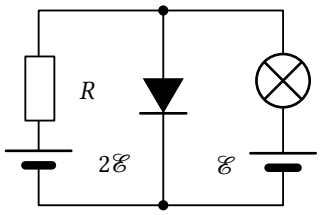


Fig. 49.1

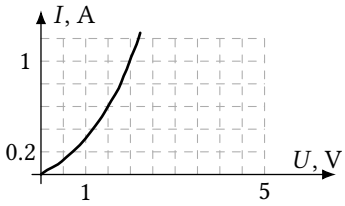


Fig. 49.2

Answer to the question The voltage across the diode for this connection equals the source voltage minus the drop on the internal resistance, i.e.

$$U_D = \mathscr{E} - Ir.$$

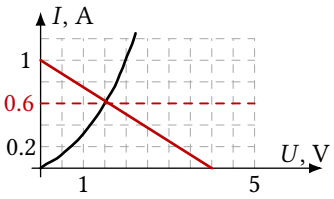


Fig. 49.3

Hence the operating current through the diode and the voltage on it are determined by the intersection of the diode's I–V curve with the source's *load line* given by

$$I = \frac{\mathscr{E} - U_D}{r}.$$

Performing the construction, we find $I_D \approx 0.6$ A.

Solution. Let U be the voltage across the diode. Choose the positive current directions in the circuit as drawn in the left-hand schematic. The dependence of the diode current on the voltage U is specified by some function; its graph is the given I–V characteristic. Denote it by $I_D = f(U)$. Then the voltage across the lamp is

$$U_\text{L} = \mathscr{E} - U,$$

and therefore

$$I_\text{L} = I_0 \cdot \sqrt{1 - \frac{U}{\mathscr{E}}}.$$

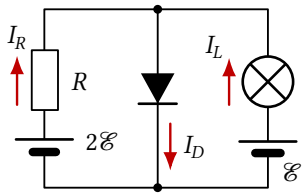


Fig. 49.4

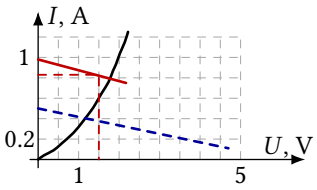


Fig. 49.5

In addition, by Ohm's law for the branch with emf $2\mathscr{E}$ we have $U = 2\mathscr{E} - I_R R \Rightarrow I_R = \dfrac{2\mathscr{E} - U}{R}$. Taking into account the current-continuity (node) equation $I_D = I_\text{L} + I_R$, we obtain

an equation for determining U:

$$f(U) = \frac{2\mathscr{E} - U}{R} + I_0 \cdot \sqrt{1 - \frac{U}{\mathscr{E}}}.$$

Since the function f is given graphically, this equation must be solved graphically. To do so we plot the graph of the right-hand side (the dashed line shows the plots of the straight line and of the parabola built for both terms separately; the solid red line is the required sum graph). Then we find the intersection point of this graph with the graph $I = f(U)$ (see the right-hand figure). As is seen, $U \approx 1.77$ V and $I_D \approx 0.81$ A. Therefore the power consumed by the diode in this circuit is

$$P_D = UI_D \approx 1.43 \text{ W}.$$

Of course, exact answers are not required. For example, the jury accepted answers within the following range:

accepted numerical range: from 1.38 W to 1.48 W; for partial credit — from 1.34 W to 1.53 W.

Remark. Of course, a graphical solution based on a "hand-drawn" graph implies some spread in answers. But in fact, to find the root with good accuracy it is sufficient to plot on the same axes three points that are easy to compute by hand; for example, at $U = 1$ V ($I \approx 0.9$ A with high accuracy and computed without a calculator), $U = 1.8$ V ($I = 0.81$ A, also computed without a calculator; moreover, this point lies very close, on the right, to the diode's I–V curve, and by itself already falls into the "narrow" acceptance interval — $P_D = 1.458$ W, which can be considered as a guaranteed upper bound for the power!), and $U = 2.5$ V ($I \approx 0.73$ A with high accuracy and also computed without a calculator by anyone who remembers $\sqrt{2}$ or $\sqrt{\frac{1}{2}}$). Drawing a smooth curve through these three points and looking at its intersection with the diode's I–V characteristic, everyone (who did this during the "testing") obtained an answer in the range from 1.40 W to 1.45 W.

P.S. The diode's I–V curve is *not* a parabola. By construction it is a section of an ellipse arc. But this is unimportant for the solution. An accurate approximation of the drawn curve by a parabola through three points leads to answers of 1.45–1.46 W, but noticeably complicates the calculations. Approximating by a parabola through two points is not quite correct.

PROBLEM 50

Linear Oscillations in an Anisotropic Quadratic Potential

From the olympiad "Pokori Vorob'evy gory!", 2024

Veronika's art
p50 . png

Question. A point mass of mass m can move in the xy-plane in a force field with potential energy

$$U(x, y) = k\,\frac{4x^2 + y^2}{2}.$$

What are the possible frequencies of its linear (i.e. along a straight line) harmonic oscillations about the equilibrium position?

Problem. On a smooth horizontal surface a ring is fixed, and the ends of three elastic bands are attached to three points on the ring; the other ends are attached to a small washer of mass $m = 250$ g. The unstretched lengths of all three bands are the same and equal to the radius of the ring. In equilibrium, the washer is such that two bands with identical stiffness coefficients $k = 1$ N/m are mutually perpendicular (see figure), while the third band has stiffness $k' = 8$ N/m and is oriented along the bisector of the angle between them. The washer is displaced by a distance $s = 1.2$ cm from this position and released from rest. It turns out that the washer moves along a straight line and returns to the equilibrium position after a time t. Find all possible values of t and the washer's speed at that moment. Note that s is much smaller than the ring radius.

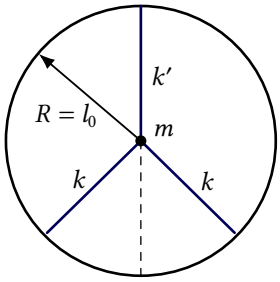


Fig. 50.1

Answer to the question If the point is displaced from equilibrium strictly along the x-axis, the change of its energy corresponds to an "ordinary" spring oscillator with oscillation frequency

$$\omega_1 = 2\sqrt{\frac{k}{m}}.$$

Similarly, for oscillations strictly along the y-axis the frequency of the resulting oscillations is

$$\omega_2 = \sqrt{\frac{k}{m}}.$$

For displacements in other directions, oscillations will occur along both axes, i.e. the law of motion will be a combination of functions with both of these frequencies.

Solution. Analysis of the participants' solutions shows that they used two fundamentally different interpretations of the statement:

(1) The bands are elastic bodies obeying Hooke's law both in tension and in compression.

(2) The bands obey Hooke's law only in tension, while in compression they slacken and do not affect the motion of the washer.

Of course, the second variant is more reasonable from the physical point of view. However, since the statement contains no explicit indication of the bands' properties other than their elasticity, the authors considered it acceptable to provide a correct solution within either interpretation.

1. We begin with the somewhat simpler *variant 1*; in this case, for greater naturalness, we replace the word "bands" by "springs".

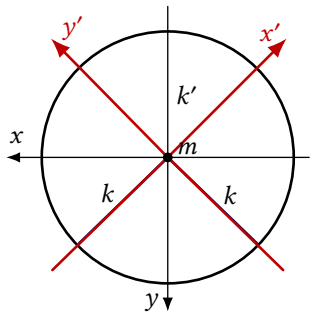


Fig. 50.2

Introduce a coordinate system (x, y) as shown in the figure. When studying the washer motion near equilibrium (since the displacements are small, according to the condition) we may restrict ourselves to terms linear in x and y in the expressions for the deformations. In this approximation the spring deformations are

$$\Delta l_1 \approx -\frac{x+y}{\sqrt{2}}, \qquad \Delta l_2 \approx y, \qquad \Delta l_3 \approx \frac{x-y}{\sqrt{2}}.$$

Therefore, the potential energy as a function of the system coordinates is described by

$$U(x,y) = \frac{k(\Delta l_1)^2}{2} + \frac{k'(\Delta l_2)^2}{2} + \frac{k(\Delta l_3)^2}{2} \approx \frac{kx^2}{2} + \frac{(k+k')y^2}{2},$$

which agrees with the answer to the question above: the motion along a straight line after a

small displacement is approximately harmonic only if the displacement is made along one of the introduced axes.

For harmonic motion, the transition from an extremal displacement to the equilibrium position occurs in one quarter of the oscillation period; hence the possible times are

$$t_1 = \frac{\pi}{2\omega_1} = \frac{\pi}{2}\sqrt{\frac{m}{k}} \approx 0.785 \text{ s}, \qquad t_2 = \frac{\pi}{2\omega_2} = \frac{\pi}{2}\sqrt{\frac{m}{k+k'}} \approx 0.262 \text{ s}.$$

In either case, when the washer passes through equilibrium its speed equals the amplitude value, and it is related to the displacement amplitude by

$$v = \omega s.$$

Thus, the possible speed values are

$$v_1 = \omega_1 s = s\sqrt{\frac{k}{m}} = 0.024 \text{ m/s}, \qquad v_2 = \omega_2 s = s\sqrt{\frac{k+k'}{m}} = 0.072 \text{ m/s}.$$

Remark. With a less successful choice of coordinate axes, an additional term that "mixes" coordinates appears in the expression for the potential energy. However, we know that for small oscillations the motion in any case can be written as a combination of oscillations along two axes with two characteristic frequencies (in theory, oscillation patterns for which the equation of motion contains harmonic functions of a single frequency are called *normal modes*). Therefore, for solving the problem in arbitrary coordinates one needs to perform a transformation from "unsuccessful" coordinates to normal ones. For example, when choosing the axes (x', y') we find that

$$\Delta l_1 \approx x', \qquad \Delta l_2 \approx -\frac{x'+y'}{\sqrt{2}}, \qquad \Delta l_3 \approx y'.$$

Then

$$U(x', y') \approx \frac{(k+k'/2)x'^2}{2} + \frac{(k+k'/2)y'^2}{2} + \frac{k'}{2}x'y'.$$

In this case, the change of variables

$$x' = -\frac{x}{\sqrt{2}} - \frac{y}{\sqrt{2}}, \qquad y' = +\frac{x}{\sqrt{2}} - \frac{y}{\sqrt{2}}$$

brings this function to the previous "convenient" form and makes it possible to carry out the same reasoning as in the "convenient" solution.

2. In this case we also use the coordinate system (x, y) and begin by constructing the expressions for the spring deformations

$$\Delta l_1 \approx -\frac{x+y}{\sqrt{2}}, \qquad \Delta l_2 \approx y, \qquad \Delta l_3 \approx \frac{x-y}{\sqrt{2}}.$$

Now we must determine which of the elastic bands "work" for each direction of displacement from the equilibrium position:

$$\Delta l_1 > 0 \text{ for } x + y < 0; \qquad \Delta l_2 > 0 \text{ for } y > 0; \qquad \Delta l_3 > 0 \text{ for } x > y.$$

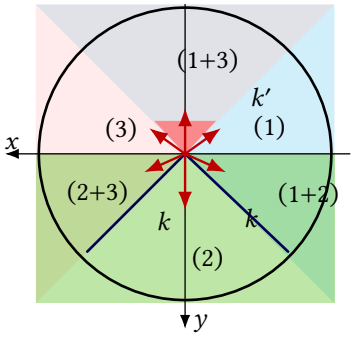


Fig. 50.3

As a result, we find which bands "work" for each displacement direction (the numbers are marked in each sector in the figure). Note that in sectors (1), (2), and (3) the washer moves under the action of the elastic force of a single band, i.e. along a straight line; moreover, this line passes through the origin only if the displacement is along the axis of the corresponding band. In sector (1+3), where the second band is slack,

$$U(x,y) = \frac{k(\Delta l_1)^2}{2} + \frac{k(\Delta l_3)^2}{2} \approx \frac{kx^2}{2} + \frac{ky^2}{2} = \frac{kr^2}{2}.$$

Therefore, the washer moves in the required way "along the radius". In these cases the equation of motion corresponds to that of a spring oscillator with stiffness k, and the time of motion to the equilibrium position is one quarter of the oscillation period of such an oscillator, i.e.

$$t_1 = \frac{\pi}{2\omega_1} = \frac{\pi}{2}\sqrt{\frac{m}{k}} \approx 0.785 \text{ s}.$$

A second variant of such motion occurs for a displacement along the y-axis; in this case the time of motion is one quarter of the oscillation period of a spring oscillator with stiffness k', i.e.

$$t_2 = \frac{\pi}{2\omega_2} = \frac{\pi}{2}\sqrt{\frac{m}{k'}} \approx 0.278 \text{ s}.$$

The corresponding speed values are

$$v_1 = \omega_1 s = s\sqrt{\frac{k}{m}} = 0.024 \text{ m/s}, \qquad v_2 = \omega_2 s = s\sqrt{\frac{k'}{m}} \approx 0.068 \text{ m/s}.$$

The most complicated case is a displacement inside the sectors (2+3) and (1+2). By symmetry, the character of the possible motion in these sectors is the same, so let us consider sector (2+3). In this case the first band is slack:

$$U(x,y) = \frac{k'(\Delta l_2)^2}{2} + \frac{k(\Delta l_3)^2}{2} \approx \frac{kx^2}{4} + \frac{(k+2k')y^2}{4} - \frac{k}{2}xy.$$

To isolate the required direction, let us pass to new variables:

$$\begin{cases} x = \tilde{x}\cos(\varphi) - \tilde{y}\sin(\varphi), \\ y = \tilde{x}\sin(\varphi) - \tilde{y}\cos(\varphi). \end{cases}$$

(From the geometric point of view this means that the new coordinate axes are obtained from the old ones by a rotation through an angle φ counterclockwise.) We choose φ so that the term $\tilde{x}\tilde{y}$ is absent in the new expression. Then we obtain

$$U(\tilde{x},\tilde{y}) \approx \frac{k\tilde{x}^2}{4}\big[1-\sin(2\varphi)\big] + \frac{k\tilde{y}^2}{4}\big[1+\sin(2\varphi)\big] + \frac{k'\tilde{x}^2}{2}\sin^2(\varphi) + \frac{k'\tilde{y}^2}{2}\cos^2(\varphi) + \frac{\tilde{x}\tilde{y}}{2}\big[k'\sin(2\varphi) - k\cos(2\varphi)\big].$$

and

$$\operatorname{tg}(2\varphi) = \frac{k}{k'} = \frac{1}{8} \Rightarrow \varphi = \frac{1}{2}\operatorname{arctg}\left(\frac{1}{8}\right),$$

that is, in sector (2+3) the $\tilde{x}$-axis lies in the required direction. For a displacement along this axis ($\tilde{y}=0$),

$$U(\tilde{x}) \approx \left[\frac{k}{2}\big(1-\sin(2\varphi)\big) + \frac{k'}{2}\big(1-\cos(2\varphi)\big)\right]\frac{\tilde{x}^2}{2} = \frac{k+k'-\sqrt{k^2+k'^2}}{2}\,\frac{\tilde{x}^2}{2}.$$

That is, the stiffness of the corresponding oscillator is

$$\tilde{k} = \frac{k+k'-\sqrt{k^2+k'^2}}{2},$$

and for this direction the motion time and speed are

$$t_3 = \frac{\pi}{2\omega_3} = \frac{\pi}{2}\sqrt{\frac{2m}{k+k'-\sqrt{k^2+k'^2}}} \approx 1.147 \text{ s},$$

$$v_3 = \omega_3 s = s\sqrt{\frac{k+k'-\sqrt{k^2+k'^2}}{2m}} \approx 0.016 \text{ m/s}.$$

ANSWERS

Answers

Short answers to all fifty problems. The full solutions start on the pages given on the right.

PART I

1 At $(x, 0)$: $E_x = -\dfrac{\sigma x}{\pi\varepsilon_0 R}$, $E_y = 0$ (toward the axis). At $(0, y)$: $E_y = \dfrac{\sigma y}{\pi\varepsilon_0 R}$, $E_x = 0$ (away from the axis). p. 6

2 $l_{\max} = \dfrac{v_0^2}{g}$; $S_{\max} = \dfrac{3\sqrt{3}\,v_0^2}{8g}$. p. 12

3 $\Delta x = \dfrac{2(\pi+1)L}{3}$. p. 15

4 $F = (\pi\rho)^{1/3}(\eta D N)^{2/3}$. p. 19

5 $v = \dfrac{2Mgs}{kS}$ (for $M \gg \rho S h$). p. 21

6 Six points: the two points where the circle touches isotherms, the two where it touches isobars, and the two where it touches isochores (lines through the origin). They divide the circle into six arcs. p. 23

7 $T_{\min}$ is just above $T_{\max} = \frac{9}{8}\,T_0$; after that the mercury is expelled by itself. p. 25

8 $H = 20$ km. An airtight balloon with the same initial mass rises less high. p. 27

9 $T = nmg$. p. 30

10 $x = A\,R\left(\dfrac{F}{ER^2}\right)^n$ with a dimensionless constant A; dimensional analysis alone does not fix n. p. 32

PART II

11 $R_A = F\,\dfrac{b}{l}$, $R_B = F\,\dfrac{a}{l}$. p. 35

12 $T = 2\pi\sqrt{\dfrac{2(R-r)}{g}}$. p. 37

13 $\lambda = \dfrac{1}{n\pi d^2} \approx 8.8\cdot 10^{-8}$ m; with the factor $1/\sqrt{2}$ for the motion of all molecules, $\approx 6.2\cdot 10^{-8}$ m. p. 40

14 $C = C_V + \dfrac{R}{2}$. p. 42

15 $p_1 = 2.33$ kPa (the vapor is saturated); $p_2 \approx 34.5$ kPa (all the water has evaporated at about 70°C). p. 44

16 The maximum temperature is $\frac{9}{8}\,T_0$, but reaching it is not enough. To expel the mercury the gas must receive $Q = \dfrac{49}{32}\,\rho g L^2 S$ (monatomic gas). p. 47

17 $T_1 = 279.4$ K; $h_1 \approx 1400$ m; $T_2 \approx 270.9$ K; about 35 mm of rain in three hours; $T_3 \approx 300$ K = 27°C, so the air at M_3 is warmer and drier than at M_0. p. 50

18 $\dfrac{dT}{dh} = -\dfrac{\gamma-1}{\gamma}\,\dfrac{Mg}{R} \approx -10$ K/km; the atmosphere would be about 30 km thick. p. 53

19 $t'/t = \frac{4}{3}$; $I_0/I = 1$ in both cases; $Q_3 = 450$ mA·h. p. 55

20 $\eta = \dfrac{16}{97} \approx 16.5\%$. p. 57

PART III

21 $h = 0$ until $t_1 \approx 4$ s; then h grows linearly at ≈ 0.08 m/s until all the water evaporates ($t_2 \approx 27$ s, $h \approx 1.7$ m); then it grows linearly at ≈ 0.2 m/s. p. 60

22 $\dfrac{\Delta m}{m} \approx \dfrac{13}{18}\dfrac{\Delta T}{T} \approx 0.0024$. p. 63

23 $v = \sqrt{v_0^2 - \dfrac{2Qq}{mR}}$ (Gaussian units, as in the solution). p. 65

24 Plate 1: $+\dfrac{Q}{2S}$ on both faces. Plate 2: $-\dfrac{Q}{2S}$ on the inner face, $+\dfrac{Q}{2S}$ on the outer face. p. 67

25 $\varphi_2 = \dfrac{q_1+q_2}{4\pi\varepsilon_0 R_2}$, $\varphi_1 = \dfrac{1}{4\pi\varepsilon_0}\left(\dfrac{q_1}{R_1}+\dfrac{q_2}{R_2}\right)$, $W = \dfrac{1}{8\pi\varepsilon_0}\left(\dfrac{q_1^2}{R_1}+\dfrac{2q_1q_2}{R_2}+\dfrac{q_2^2}{R_2}\right)$. After grounding $q_1' = -q_2\dfrac{R_1}{R_2}$ and $W' = \dfrac{q_2^2(R_2-R_1)}{8\pi\varepsilon_0 R_2^2} < W$. p. 69

26 $l > R$: $\varphi = \dfrac{1}{4\pi\varepsilon_0}\left(\dfrac{Q}{R}+\dfrac{q}{l}\right)$; $l < R$: $\varphi = \dfrac{Q+q}{4\pi\varepsilon_0 R}$, independent of l. p. 71

27 $C_{\text{eq}} \approx 26.4\,C = 26.4\ \mu$F. p. 74

28 $W_{\min} = e|U_m| = 2.25$ eV; time of flight $\tau \approx 10^{-8}$ s. p. 76

29 $\dfrac{v_{\max}}{v_{\min}} = \sqrt{\dfrac{v_1^2+v_0^2}{v_1^2-v_0^2}}$. p. 79

30 $n \approx 1.64$ (slope of $\ln F_{\text{drag}}$ versus $\ln v$). p. 81

PART IV

31 $\alpha = \arctan\frac{1}{2} \approx 26.6^\circ$; $T = 2\pi\sqrt{\dfrac{2m}{5\sigma}}$. p. 87

32 $v_{\min} = \sqrt{\sqrt{\dfrac{(\omega_0 R)^4}{4} + 4\left(\dfrac{F_0 l}{m}\right)^2} - \dfrac{(\omega_0 R)^2}{2}}$; $\omega_1 = \dfrac{\omega_0}{\sqrt{2}}$; for τ see the solution. p. 92

33 $C_{\text{sat}} = R(4-\alpha) \approx -94$ J/(mol·K); compression: $\Delta T_1 \approx +5.6$ K (the vapor becomes unsaturated); expansion: $\Delta T_2 \approx -1.2$ K (part of the vapor condenses). p. 96

34 Without the lens: a central spot of diameter d and two rings with mean radii $R_1 = \dfrac{3dL}{h}$ and $R_2 = \dfrac{6dL}{h}$. With the lens: ring radii $\dfrac{3dL}{2h}$ and $\dfrac{3dL}{h}$; the spot and the ring widths become much smaller than d. p. 99

35 (1) Ammeter A_4. (2) $I_0 = \dfrac{8I_1}{3}$. (3) $I_{A2} = \dfrac{I_0}{3}$ for $I_0 < I_1$ and $I_{A2} = \dfrac{8I_1}{15} - \dfrac{I_0}{5}$ for $I_0 > I_1$. p. 103

36 A construction problem: the lens is converging, and there are two cases (real and virtual image of B). See the solution for the constructions. p. 107

37 $R = (10.7 \pm 1.6)$ cm. p. 110

38 $R/r \approx 3.86$; $T \approx 5900$ min; $\rho \approx 7.9 \cdot 10^3$ kg/m^3. p. 113

39 If $v_0 \le 2a\tau$: $v_{\min} = a\tau\cot\alpha$, $v_{\max} = a\tau\sqrt{\cot^2\alpha + 4\cot\alpha + 8}$. If $v_0 \ge 2a\tau$: $v_{\min} = 2a\tau(\cot\alpha - 1)$, $v_{\max} = \dfrac{2a\tau}{\sin\alpha}$. p. 117

40 $\eta_{\max} = \dfrac{1}{8}$. p. 120

PART V

41 $L_{\min} = \dfrac{2kR^2}{kR - 2mg}$. p. 123

42 $r = \dfrac{R - l\left(\dfrac{R}{F} - \tan\alpha\right)}{\sqrt{1 + \left(\dfrac{R}{F} - \tan\alpha\right)^2}}$. p. 125

43 $2\alpha = 180^\circ - 4\varphi$, where $\cos\varphi = \dfrac{\sin\alpha_0 + \sqrt{\sin^2\alpha_0 + 8}}{4}$. p. 130

44 $T_{N+1} = \dfrac{NP}{(N+1)P_0 - NP}\left(\sum\limits_{i=1}^{N}\dfrac{1}{T_i}\right)^{-1}$ if $(N+1)P_0 > NP$; otherwise there is no solution. p. 133

45 If $a = \dfrac{mg}{kL} \le 2$: the upper piston stays, the lower one goes down by $\dfrac{mg}{2k}$. If $a > 2$: the lower piston rests on the bottom, the upper one is at height $L\,\dfrac{\sqrt{a^2 + 10a + 1} + 1 - a}{2}$. p. 135

46 $F = \dfrac{3kq^2}{32R^2}$, perpendicular to the base plane of the larger hemisphere. p. 138

47 $t_1 = \sqrt{\frac{3}{4}}\,t \approx 0.87\,t$. p. 142

48 $H_{\max} = H\,\dfrac{\sin^2\left(\frac{10\pi + 2\alpha}{22}\right)}{\sin^2\alpha}$. p. 144

49 Question: $I \approx 0.6$ A. Problem: $P_D \approx 1.43$ W. p. 146

50 Question: $\omega_1 = 2\sqrt{k/m}$ (along x), $\omega_2 = \sqrt{k/m}$ (along y). Problem: if the bands obey Hooke's law also in compression, $t \approx 0.785$ s or 0.262 s, $v \approx 0.024$ or 0.072 m/s; if they slacken in compression, $t \approx 0.785$, 0.278 or 1.147 s, $v \approx 0.024$, 0.068 or 0.016 m/s. p. 149

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ABOUT THE AUTHORS

About the authors

Veronika's art
ivan.jpg

Ivan Rudik

[A short paragraph about Ivan: olympiad background, studies, why he wrote this book.]

Veronika's art
veronika.jpg

Veronika Lukanovskaya

[A short paragraph about Veronika and her illustrations.]

SOURCES

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Where each problem comes from. Titles of Russian articles are transliterated, with a translation in brackets.

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